Golden

MATHS SERIES

LINEAR ALGEBRA

CONTENTS

SYLLABUS

Definitions of groups, rings and fields with illustrations.

Vector space, linear dependence and independence, basis, Dimension, Row and Column spaces of matrices, Connection with rank. The space of solution of a system of linear homogeneous equations.

Linear transformations and their algebra, connection between matrices and linear transformations. Determinant and trace of a linear transformation. Rank and nullity of a linear transformation. Eigen values and Eigen vectors. Cayley-Hamilton Theorem.

"Some special types of matrices : Orthogonal, symmetric, skew-symmetric, unitary ; Hermitian and skew-Hermitian matrices", their eigen values.

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BASIC CONCEPTS

1. Sets.

(*a*) **Def.** *A set is a well defined collection of objects,* which are called **members** *or* **elements** *or* **points** *of the set.*

For Examples : (*i*) The collection N is the set of all natural numbers.

(*ii*) The collection Z is the set of all integers.

(*iii*) The collection Q is the set of all rational numbers.

(*iv*) The collection R is the set of all real numbers.

(*v*) The collection C is the set of all complex numbers.

If S be a set and x be an element, then

(*i*) $x \in S$ means "*x belongs to S*"

(*ii*) $x \notin S$ means "*x does not belong to S*".

Notations. The sets are usually denoted by capital Roman letters and the elements are usually by small Roman or Greek letters.

(*b*) **Representation.** A set is represented either by describing all its elements or by stating the property which determines whether an element belongs to S or not.

For Examples : (*i*) If A has integers 1, 2, 3, 4, 5, then we write A = {1, 2, 3, 4, 5}.

(*ii*) If B is the set of those integers which are squares of some other integers, then B has integers 0, 1, 4, 9, 16, 25,……

We write $B = \{x : x \text{ is a square of some integer}\}$.

In general, $A = \{x : P(x)\}$ means a set.

A consists of those elements, which satisfy the property P.

(*c*) **Subset. Def.** *A set A is said to be a subset of a set B if every element of A is also in B.*

For Examples : (*i*) The set Z of all integers is a subset of Q, the set of all rational numbers.

(*ii*) The set Q of all rational numbers is a subset of R, the set of all real numbers.

(*d*) **Null Set. Def.** *A set having no element is said to be a null set.*

It is also known as **Empty set** or **Void set.**

Null set is usually denoted by ϕ.

For Examples : (*i*) $A = \{x : x > 0 \text{ and } x < 0\}$ is ϕ.

(*ii*) $B = \{x : x \in R \text{ and } x^2 + 1 = 0\}$ is ϕ.

(*e*) **Union and Intersection.**

(*i*) **Union. Def.** *(A ∪ B) is the set* **(Union of A and B)**, *which consists of all those elements which are either in A or in B or in both.*

For Examples : (*i*) Let A = {1, 3, 5, 9} and B = {3, 5, 10}, then A ∪ B = {1, 3, 5, 9, 10}.

(*ii*) Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8}, then

A ∪ B = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

(*ii*) **Intersection. Def.** *(A ∩ B) is the set* **(Intersection of A and B)**, *which consists of all those elements which belong to A as well as B.*

For Examples : (*i*) Let A = {1, 3, 5, 9} and B = {3, 5, 10}, then A ∩ B = {3, 5}.

(*ii*) Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8}, then A ∩ B = ϕ.

(*f*) **Cartesian Product.**

Given two objects a and b, we form an object (a, b), called an ordered pair of a and b, with a as first co-ordinate and b as second co-ordinate such that

$(a, b) = (c, d)$ holds iff $a = c$ and $b = d$.

Thus (1, 2) and (2, 1) are not same.

Def. *Given two sets A and B. The set consisting of all ordered pairs (a, b) with $a \in A$ and $b \in B$, is called the cartesian product of A and B and is denoted by $A \times B$.*

For Examples : Let $A = \{1, 2, 3\}$ and $B = \{1, 6\}$. Then

(*i*) $A \times B = \{(1, 1), (1, 6), (2, 1), (2, 6), (3, 1), (3, 6)\}$; and

(*ii*) $B \times A = \{(1, 1), (1, 2), (1, 3), (6, 1), (6, 2), (6, 3)\}$.

2. Functions.

Def. *A function f from a set A to a set B is a rule or law which associates each $a \in A$ to a unique element $b \in B$.*

This is also known as **mapping**.

$f : A \to B$ means that "f *is a function from A to B*".

Here b is called the **image** of a under f and, we write $b = f(a)$.

The set $A = \{ a, f(a) : a \in A\}$ is a subset of $A \times B$ is called the **graph** of f.

Other Definitions :

(*i*) The set $\{ a : (a, b) \in S$ for some $b\}$ is called the **domain** of S.

(*ii*) The set $\{ b : (a, b) \in S$ for some $a\}$ is called the **range** of S.

(*iii*) The function $f : A \to B$ is said to be **one-one** if for any a_1, a_2

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

Or

$$f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2.$$

(*iv*) The function $f : A \to B$ is said to be **onto** if R_f (Range of f) = B.

A function, which is one-one onto, is called a **bijection.**

3. Binary Compositions.

See Art 1 ; Chapter 2.

4. Group.

(*a*) Let us deal with operations of addition and multiplication among numbers.

(I) *Consider Z, the set of all integers.*

We know that for any two integers a and b, the sum $a + b$ is also an integer.

Given an ordered pair (a, b) of the elements of Z, addition (+) determines a unique element $a + b$ of Z. Thus we say that *Z is closed under addition.*

Other Properties of Addition

(*i*) **Associativity.** $(a + b) + c = a + (b + c) \; \forall \; a, b, c \in Z$.

(*ii*) **Existence of Identity.** There exists $0 \in Z$ such that $a + 0 = a = 0 + a \; \forall \; a \in Z$.

Here 0 is called the **additive identity** or **zero element** of Z.

(*iii*) **Existence of Inverse.** There exists $a' \in Z$ such that

$$a + a' = 0 = a' + a \; \forall \; a \in Z.$$

Here a' $(= -a)$ is called the **additive inverse** or **negative** of a.

(*iv*) **Commutativity.** $a + b = b + a \; \forall \; a, b \in Z$.

(II) *Consider Q^*, the set of all non-zero rational numbers.*

Here Q^* is closed under multiplication.

[$\because$ *Given an ordered pair (a, b) of the elements of Q^*, multiplication (.) determines a unique element a . b of Q^**]

Other Properties of Multiplication

(*i*) **Associativity.** $(a \,.\, b).\, c = a.\,(b \,.\, c)\ \forall\ a, b, c \in Q^*$.

(*ii*) **Existence of Identity.** There exists $1 \in Q^*$ such that $a \,.\, 1 = a = 1 \,.\, a\ \forall\ a \in Q^*$.

Here 1 is called the **multiplicative identity** or **unit element** of Q^*.

(*iii*) **Existence of Inverse.** There exists $a' \in Q^*$ such that $a \,.\, a' = 1 = a' \,.\, a\ \forall\ a \in Q^*$.

Here a' $(= a^{-1})$ is called the **multiplicative inverse** or **reciprocal** of a.

(*iv*) **Commutativity.** $a \,.\, b = b \,.\, a\ \forall\ a, b \in Q^*$.

(*b*) **1. Group.**

A system $< G, * >$, where G is non-empty set and $*$ is a binary composition on G, is called a group if it satisfies the following postulates : **(G.N.D.U. 1997)**

(*i*) **Closure Axiom :** $\forall\ a, b \in G \Rightarrow a * b \in G$.

(*ii*) **Associative Law :** $a * (b * c) = (a * b) * c\ \forall\ a, b, c \in G$.

(*iii*) **Existence of Identity :** *There exists an element $e \in G$, called an* **identity**, *such that $a * e = a = e * a\ \forall\ a \in G$.*

(*iv*) **Existence of Inverse :** $\forall\ a \in G$, *there exists an element $a^{-1} \in G$, called the* **inverse** *of a, such that $a * a^{-1} = e = a^{-1} * a$.*

Caution. a^{-1} does not mean $\frac{1}{a}$.

2. Commutative Group or Abelian Group.

If in addition to the above four postulates, the following postulate is also satisfied, the group G is called a **Commutative or an Abelian group.**

(*v*) **Commutative Law.** $\forall\ a, b \in G,\ a * b = b * a$.

3. Non-Commutative Group or Non-abelian Group.

If the group does not satisfy the above postulate (v), then the group G is called Non-commutative or Non-abelian group.

4. Finite and Infinite Group.

If the number of elements in the group G is finite, then $< G, * >$ is called a finite group, otherwise it is called an infinite group.

5. Order of the group.

The number of elements in a finite group is called the order of the group.

This is denoted by O (G) or | G |.

The infinite group is of infinite order.

6. Semi-Group.

A system $<G, *>$, where G is a non-empty set and $*$ is a binary composition on G, is called a semi-group if it satisfies the following postulates :

(*i*) **Closure Axiom :** $\forall\ a, b \in G \Rightarrow a * b \in G$.

(*ii*) **Associative Law :** $a * (b * c) = (a * b) * c\ \forall\ a, b, c \in G$.

Conclusion : Every group is a semi-group but every semi-group may or may not be a group.

SOLVED EXAMPLES

Example 1. *Prove that $< Z, + >$, where Z is the set of all integers, is an infinite abelian group.*

Sol. The system is $< Z, +>$, where

$Z = \{......-3, -2, -1, 0, 1, 2, 3,......\}$ and '+' is the binary composition in Z.

(*i*) **Closure Axiom.** $\forall\, a, b \in Z \Rightarrow a + b \in Z.$

[$\because$ *Sum of any two integers is an integer*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c \;\forall\, a, b, c \in Z.$

(*iii*) **Existence of Identity.** There exists an element $0 \in Z$, such that $a + 0 = a = 0 + a \;\forall\, a \in Z.$

(*iv*) **Existence of Inverse.** $\forall\, a \in Z$, there exists an element $-a \in Z$, such that $a + (-a) = 0 = (-a) + a.$

Remember : *$-a$ is the inverse of a under addition. Thus $< Z, + >$ is a group.*

(*v*) **Commutative Law.** $\forall\, a, b \in Z, a + b = b + a.$

Thus $< Z, + >$ is an abelian group.

(*vi*) Since the number of integers is infinite,

$\therefore$ Z is an infinite set.

Hence $< Z, + >$ is an infinite abelian group.

Example 2. *Prove that $< Q, + >$, where Q is a set of rational numbers, is an infinite abelian group.*

Sol. The system is $< Q, + >$, where Q is the set of rational numbers and '+' is the binary operation in Q.

(*i*) **Closure Axiom.** $\forall\, a, b \in Q \Rightarrow a + b \in Q.$

[$\because$ *Sum of any two rational numbers is a rational number*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c \;\forall\, a, b, c \in Q.$

(*iii*) **Existence of Identity.** There exists an element $0 \in Q$ such that $a + 0 = a = 0 + a \;\forall\, a \in Q.$

(*iv*) **Existence of Inverse.** $\forall\, a \in Q$, there exists an element $-a \in Q$ such that $a + (-a) = 0 = (-a) + a.$
Thus $< Q, + >$ is a group.

(*v*) **Commutative Law.** $\forall\, a, b \in Q, a + b = b + a.$
Thus $< Q, + >$ is an abelian group.

(*vi*) Since the number of rational numbers is infinite,

$\therefore$ Q is an infinite set.

Hence $< Q , + >$ is an infinite abelian group.

Example 3. *Prove that $<R, +>$, where R is the set of real numbers is an infinite abelian group.*

Sol. Exactly similar to Ex. 2. [Replace Q by R]

Example 4. *Prove that $< C, + >$, where C is the set of complex numbers is an infinite abelian group.*

Sol. The system is $<C, +>$, where C is the set of complex numbers and '+' is the binary operation in C.

(*i*) **Closure Axiom.** $\forall\, a, b \in C \Rightarrow a + b \in C.$

[$\because$ *Sum of any two complex numbers is a complex number*]

(*ii*) **Associate Law.** $a + (b + c) = (a + b) + c \;\forall\, a, b, c \in C.$

(*iii*) **Existence of Identity.** There exists an element $0 \in C$ such that $a + 0 = a = 0 + a \;\forall\, a \in C.$

[**Remember :** 0 is a complex number because $0 = 0 + i\,(0)$]

(*iv*) **Existence of Inverse.** $\forall\, a \in C$, there exists an element $-a \in C$ such that $a + (-a) = 0 = (-a) + a.$
Thus $<C, + >$ is a group.

(*v*) **Commutative Law.** $\forall\, a, b \in C, a + b = b + a.$
Thus $< C, + >$ is an abelian group.

(*vi*) Since the number of complex numbers is infinite,

$\therefore$ C is an infinite set.

Hence $<C, +>$ an infinite abelian group.

Example 5. *Prove that $< N , +>$, where N is a set of natural numbers, is a semi-group and not a group.*

Sol. The system is $< N, +>$, where N is a set of natural numbers and '+' is the binary operation in N.

(*i*) **Closure Axiom.** $\forall\ a, b \in N \Rightarrow a + b \in N$.

[$\because$ *Sum of any two natural numbers is a natural number*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c\ \forall\ a, b, c \in N$.

Thus $<N, + >$ is a semi-group.

(*iii*) **Existence of Identity.** Under addition composition, 0 is the identity element.

But $0 \notin N$. [$\because$ *0 is not a natural number*]

Thus $<N , +>$ is not a group.

Hence $<N, + >$ is a semi-group and not a group.

Example 6. *Prove that $< N', + >$, where $N' = \{0, 1, 2, 3, ...\}$ is a semi-group and not a group.*

Sol. The system is $<N' + >$, where $N' = \{0, 1, 2, 3, ...\}$ and '+' is the binary operation in N'.

(*i*) **Closure Axiom.** $\forall\ a, b \in N' \Rightarrow a + b \in N'$.

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c\ \forall\ a, b, c \in N'$.

Thus $< N', + >$ is a semi-group.

(*iii*) **Existence of Identity.** There exists an element $0 \in N'$, such that $a + 0 = a = 0 + a\ \forall\ a \in N'$.

(*iv*) **Existence of Inverse.** Under addition composition, $-a$ is the inverse of a.

But $-a \notin N'$. [$\because$ *N' contains no –ve integer*]

Thus $< N', +>$ is not a group.

Hence $<N', +>$ is a semi-group and not a group.

Example 7. *Prove that $<N, \times>$, where N is a set of natural numbers is a semi-group with an identity element.*

Sol. The system is $<N, \times>$, where N is a set of natural numbers and '×' is the binary operation in N.

(*i*) **Closure Axiom.** $\forall\ a , b \in N \Rightarrow a \times b \in N$.

[$\because$ *Product of any two natural numbers is a natural number*]

(*ii*) **Associative Law.** $a \times (b \times c) = (a \times b) \times c\ \forall\ a, b, c \in N$.

Thus $<N, \times>$ is a semi-group.

(*iii*) **Existence of Identity.** There exists an element $1 \in N$ such that $a \times 1 = a = 1 \times a\ \forall\ a \in N$.

Thus $<N, \times>$ is with an identity element.

Hence $<N, \times>$ is a semi-group with an identity element.

Example 8. *Prove that $<Q, \times>$ is not a group, where Q is a set of rational numbers.*

Sol. The system $< Q, \times >$, where Q is a set of rational numbers.

(*i*) **Closure Axiom.** $\forall\ a, b \in Q,\ a . b \in Q$.

[$\because$ *Product of two rational numbers is a rational number*] (*ii*) **Associative Law.** $a. (b. c) = (a. b) . c \; \forall \; a, b, c \in Q$.

(*iii*) **Existence of Identity.** There exists an element $1 \in Q$ such that $a . 1 = a = 1 . a \; \forall \; a \in Q$.

(*iv*) **Existence of Inverse.** There is no multiplicative inverse of $0 \in Q$.

Thus 0 has no inverse under multiplication.

Hence $< Q, \times >$ is not a group.

Example 9. *Prove that $<S, \times>$, where $S = \{1\}$ is a finite abelian group.*

Sol. The system is $< S, \times >$, where $S = \{1\}$ and '$\times$' is the binary operation in S.

The element 1 can be repeated again and again.

It is closed, associative law holds.

The identity element 1 exists.

The inverse of 1 is 1, which is in S.

Also S contains only one element.

Hence $<S, \times>$, where $S = \{1\}$ is a finite abelian group.

Example 10. *Prove that the set of non-zero real numbers from a group under multiplication.*

Sol. The system is $<R', \times>$, where R' is the set of all non-zero real numbers.

(*i*) **Closure Axiom.** $\forall \; a, b \in R' \;\; \Rightarrow \;\; a \times b \in R'$.

[$\because$ *Product of two non-zero reals is a non-zero real*]

(*ii*) **Associative Law.** $a \times (b \times c) = (a \times b) \times c \; \forall \; a, b, c \in R'$.

(*iii*) **Existence of Identity.** There exists an element $1 \in R'$ such that $a \times 1 = a = 1 \times a \; \forall \; a \in R'$.

(*iv*) **Existence of Inverse.** $\forall \; a \in R'$, where $a \neq 0$, there exists $\dfrac{1}{a} \in R'$ such that

$$a . \frac{1}{a} = 1 = \frac{1}{a} . a.$$

Hence the set of non-zero real numbers form a group under multiplication.

Example 11. *Prove that the set of non-zero complex numbers form a group under multiplication.*

Sol. The system is $<C', \times>$, where C' is the set of all non-zero complex numbers.

Here $C' = \{x + i\,y \mid x, y \text{ are not both zero and } x, y \in R\}$.

(*i*) **Closure Axiom.** Let $z_1 = a + ib$ and $z_2 = c + id \in C'$.

Then $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i\,(ad + bc)$.

Now $z_1 z_2 = 0$ if $ac - bd = 0$ and $ad + bc = 0$

if $a = 0 = b$ or $c = 0 = d$

if either $a + ib$ or $c + id$ is a zero complex number.

But $a + ib$ and $c + id$ are both non-zero complex numbers.

Thus $z_1 z_2$ is a non-zero complex number.

$\Rightarrow \;\; z_1 z_2 \in C'$.

(*ii*) **Associative Law.**

Let $z_1 = a + ib, z_2 = c + id$ and $z_3 = e + if \in C'$.

Then $(z_1 z_2) z_3 = [(a + ib)(c + id)](e + if) = [(ac - bd) + i(ad + bc)](e + if)$

$= [(ac - bd)e - (ad + bc)f] + i[(ac - bd)f + (ad + bc)e]$

Also $z_1(z_2 z_3) = (a + ib)[(c + id)(e + if)] = (a + ib)[(ce - df) + i(cf + de)]$

$= [a(ce - df) - b(cf + de)] + i[b(ce - df) + a(cf + de)]$

$= [(ac - bd)e - (ad + bc)f] + i[(ac - bd)f + (ad + bc)e]$

$\therefore \quad (z_1 z_2) z_3 = z_1(z_2 z_3)$.

Thus C′ is associative.

(*iii*) **Existence of Identity.**

For $a + ib \in C'$, there exists $1 + i0 \in C'$ such that

$(a + ib)(1 + i0) = a + ib = (1 + i0)(a + ib)$

Thus $1 + i0$ is the identity element.

(*iv*) **Existence of Inverse.**

Let $z = a + ib \in C'$, where $a, b \in R$ and a, b are not both zero.

Now $\dfrac{1}{z} = \dfrac{1}{a+ib} = \dfrac{a-ib}{(a+ib)(a-ib)} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} + i\left(\dfrac{-b}{a^2+b^2}\right)$.

Since a, b are both non-zero, therefore, $a^2 + b^2 \neq 0$.

Thus $\dfrac{1}{z} \in C'$

$\therefore \quad \dfrac{a}{a^2+b^2} + i\left(\dfrac{-b}{a^2+b^2}\right)$ is the multiplicative inverse of $a + ib$.

Hence C′ is a group under multiplication.

Example 12. *Show that the set of all even integers (including zero) is an abelian group under addition.* (P.U. 1990)

Sol. The system is $\langle E, + \rangle$, where E is the set of all even integers (including zero).

(*i*) **Closure Axiom.** $\forall\, a, b \in E \Rightarrow a + b \in E$.

[$\because$ *Sum of two even integers (including zero) is an even integer*]

(*ii*) **Associative Law.**

Let $x = 2k_1, y = 2k_2$ and $z = 2k_3$, where $k_1, k_2, k_3 \in Z$.

Now $\quad (x + y) + z = (2k_1 + 2k_2) + 2k_3 = 2(k_1 + k_2) + 2k_3 = 2[(k_1 + k_2) + k_3]$

$= 2[k_1 + (k_2 + k_3)]$ [$\because$ *Associative Property holds in Z*]

$= 2k_1 + 2(k_2 + k_3) = 2k_1 + (2k_2 + 2k_3) = x + (y + z)$.

Thus associative law holds in E.

(*iii*) **Existence of Identity.**

For each $x = 2k \in E$, there is $0 = 2(0) \in E$ such that

$x + 0 = 2k + 2(0) = 2(k + 0) = 2k = x$

and $\quad 0 + x = 2(0) + 2k = 2(0 + k) = 2k = x$

so that $\quad x + 0 = x = 0 + x$.

Thus 0 is the identity element in E.

(*iv*) **Existence of Inverse.**

For each $x = 2k \in E$, there is $-x = -2k = 2(-k) \in E$

such that $x + (-x) = 2k + (-2k) = 2\left(k + (-k)\right) = 2(0) = 0$

and $(-x) + x = (-2k) + 2k = 2\left((-k) + k\right) = 2(0) = 0.$

Thus each element of E possesses its inverse.

(*v*) **Commutative Law.**

Let $x = 2k_1$ and $y = 2k_2 \in E$, where $k_1, k_2 \in Z$.

Then $x + y = 2k_1 + 2k_2 = 2(k_1 + k_2) = 2(k_2 + k_1) = 2k_2 + 2k_1 = y \in E$

Thus $x + y = y + x \ \forall x, y \in E$.

Hence E is an abelian group under addition.

Example 13. *Prove that $<S, \times>$, where $S = \{1, -1\}$ is a finite abelian group.*

Sol. The given system is $<S, \times>$, where $S = \{1, -1\}$.

(*i*) **Closure Axiom.**

Since $1 \times (-1) = -1 \in S$,

$\therefore$ S is closed under '$\times$'.

(*ii*) **Associative Law** is obvious.

(*iii*) **Existence of Identity.**

Here 1 is the identity element.

(*iv*) **Existence of Inverse.**

1 is the inverse of 1 and -1 is the inverse of -1.

(*v*) **Commutative Law.**

Here $1 \times (-1) = (-1) \times 1$.

Thus commutative law holds in S.

Hence $<S, \times>$ is a finite abelian group.

Example 14. *Prove that $<S, \times>$, where $S = \{1, \omega, \omega^2\}$, where $1, \omega, \omega^2$ are cube roots of unity, is a finite abelian group.* (Pbi. U. 1996)

Sol. The system is $<S, \times>$, where

$S = \{1, \omega, \omega^2\}$ while $1, \omega, \omega^2$ are cube roots of unity and thus $\omega^3 = 1$, and '$\times$' is the binary operation is S.

(*i*) **Closure Axiom.**

Since $1 \times \omega = \omega \in S$, $1 \times \omega^2 = \omega^2 \in S$

and $\omega \times \omega^2 = \omega^3 = 1 \in S$,

$\therefore$ S is closed under '$\times$'.

(*ii*) **Associative Law.**

$$1 \times (\omega \times \omega^2) = 1 \times (\omega^3) = 1 \times 1 = 1$$

and $$(1 \times \omega) \times \omega^2 = \omega \times \omega^2 = \omega^3 = 1.$$

Thus $1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$.

$\therefore$ Associative Law holds.

(*iii*) **Existence of Identity.**

Under multiplication 1 works for identity and $1 \in S$

$\therefore$ Identity element *i.e.*, 1 exists.

(*iv*) **Existence of Inverse.**

Since $1 \times 1 = 1 = 1 \times 1$, $\therefore$ 1 is the inverse of 1.

Since $\omega \times \omega^2 = 1 = \omega^2 \times \omega$, $[\because \omega^3 = 1]$

$\therefore$ ω^2 is the inverse of ω.

Since $\omega^2 \times \omega = 1 = \omega \times \omega^2$, $[\because \omega^3 = 1]$

$\therefore$ ω is the inverse of ω^2.

$\therefore$ Inverse of each element of S exists.

Thus < S, ×> is a group.

(*v*) **Commutative Law.**

Now $1 \times \omega = \omega \times 1$, $[\because \text{ each} = \omega]$

$1 \times \omega^2 = \omega^2 \times 1$ $[\because \text{ each} = \omega^2]$

and $\omega \times \omega^2 = \omega^2 \times \omega$ $[\because \text{ each} = \omega^3 = 1]$

$\therefore$ Commutative Law holds.

Thus < S, ×> is an abelian group.

(*vi*) Since S contains three elements, $\therefore$ S is finite.

Hence <S, ×> is a finite abelian group.

Example 15. *Prove that < S, ×>, where S is a set of 4th roots of unity i.e., S = {1, – 1, i, – i }, is a group, where* $i^2 = -1$.

Sol. The system is < S, ×>, where S = $\{1, -1, i, -i\}$ and '×' is the binary operation in S.

(*i*) **Closure Axiom.**

Since $1 \times (-1) = -1 \in S$, $1 \times i = i \in S$,

$1 \times (-i) = -i \in S$, $(-1) \times i = -i \in S$,

$(-1) \times (-i) = i \in S$, $i \times (-i) = -i^2 = 1 \in S$.

$\therefore$ S is closed under '×'.

(*ii*) **Associative Law.**

$1 \times (-1 \times i) = 1 \times (-i) = -i$,

$\big(1 \times (-1)\big) \times i = (-1) \times i = -i$,

$\therefore$ $1 \times (-1 \times i) = \big(1 \times (-1)\big) \times i$.

Similarly with any other three members of S.

$\therefore$ Associative Law holds.

(*iii*) **Existence of Identity.**

Under multiplication 1 works for identity and $1 \in S$.

$\therefore$ Identity element *i.e.*, 1 exists.

(*iv*) **Existence of Inverse.**

Since $1 \times 1 = 1 = 1 \times 1$, $\therefore$ 1 is the inverse of 1

$(-1) \times (-1) = 1 = (-1) \times (-1)$, $\therefore$ -1 is the inverse of -1

$i \times (-i) = 1 = (-i) \times i$, $\therefore$ $-i$ is the inverse of i

and $-i \times i = 1 = i \times (-i)$, $\therefore$ i is the inverse of $-i$.

$\therefore$ Inverse of each element of S exists.

Hence <S, ×> is a group.

Example 16. *Prove that* Q^+, *the set of positive rational numbers is an abelian group under the operation defined as* $a * b = \dfrac{ab}{3}$ $\forall\, a, b \in Q^+$.

Sol. (*i*) **Closure Axiom.**

$$\forall a, b \in Q^+ \Rightarrow ab \in Q^+ \Rightarrow \frac{ab}{3} \in Q^+$$

$\Rightarrow \quad a * b \in Q^+$

Thus Q^+ is closed under '$*$'.

(*ii*) **Associative Law.**

Let $a, b, c \in Q^+$.

Now $a * (b * c) = a * \left(\frac{bc}{3}\right) = \dfrac{a\left(\frac{bc}{3}\right)}{3} = \dfrac{abc}{9}$.

And $(a * b) * c = \left(\frac{ab}{3}\right) * c = \dfrac{\left(\frac{ab}{3}\right)c}{3} = \dfrac{abc}{9}$.

$\therefore \quad a * (b * c) = (a * b) * c$.

Thus Associative Law holds under '$*$'.

(*iii*) **Existence of Identity.**

To Prove : $\forall a \in Q^+$, there exists $e \in Q^+$ such that $a * e = a = e * a$.

Now $a * e = a \Rightarrow \dfrac{ae}{3} = a$

$\Rightarrow \quad \dfrac{ae}{3} - a = 0 \quad \Rightarrow \dfrac{a}{3}(e-3) = 0$

$\Rightarrow \quad e = 3.$ $\qquad [\because \ a \neq 0 \text{ as } a \in Q^+]$

Similarly $e * a = a \quad \Rightarrow e = 3$.

Thus Q^+ possesses identity element $3 \in Q^+$.

(*iv*) **Existence of Inverse.**

To Prove : $\forall a \in Q^+$, there exists $x \in Q^+$ such that $a * x = 3 = x * a$.

Now $\quad a * x = 3 \quad \Rightarrow \dfrac{ax}{3} = 3 \qquad \Rightarrow x = \dfrac{9}{a}$

Similarly $x * a = 3 \qquad \Rightarrow x = \dfrac{9}{a}$.

Thus Q^+ possesses an inverse element $\dfrac{9}{a} \in Q^+$ of $a \in Q^+$.

(*v*) **Commutative Law.**

$\forall a, b \in Q^+, a * b = \dfrac{ab}{3} = \dfrac{ba}{3} = b * a$.

Hence $\langle Q^+, * \rangle$ is an abelian group.

Example 17. *Prove that the set of all rational numbers of the type $2^a 3^b$ $(a, b \in Z)$ is a group with respect to multiplication of rationals.*

Sol. We have $G = \{ x \mid x = 2^a 3^b ; a, b \in Z \}$.

(*i*) **Closure Axiom.**

Let $x = 2^a 3^b$ and $y = 2^c 3^d$, where $a, b, c, d \in Z$.

Now $x y = (2^a 3^b)(2^c 3^d) = (2^a 2^c)(3^b 3^d) = 2^{a+c} 3^{b+d}$

$\in G$. [$\because$ *$a + c$ and $b + d \in Z$ as $a, b, c, d \in Z$*]

Thus G is closed under multiplication.

(*ii*) **Associative Law.**

Let $x = 2^a 3^b, y = 2^c 3^d$ and $z = 2^e 3^f$, where $a, b, c, d, e, f \in Z$.

Now $x(yz) = (2^a 3^b)[(2^c 3^d)(2^e 3^f)] = (2^a 3^b)[(2^c 2^e)(3^d 3^f)] = (2^a 3^b)[(2^{c+e})(3^{d+f})]$

$= (2^a 2^{c+e})(3^b 3^{d+f}) = 2^{a+(c+e)} 3^{b+(d+f)}$.

Similarly $(xy)z = 2^{a+(c+e)} 3^{b+(d+f)}$.

$\therefore$ $x(yz) = (xy)z \quad \forall x, y, z \in G$.

Thus Associative Law holds in G.

(*iii*) **Existence of Identity.**

For each $x = 2^a 3^b \in G$, there is $1 = 2^0 3^0 \in G$ such that

$x \cdot 1 = (2^a 3^b)(2^0 3^0) = 2^{a+0} 3^{b+0} = 2^a 3^b = x$

and $1 \cdot x = (2^0 3^0)(2^a 3^b) = 2^{0+a} 3^{0+b} = 2^a 3^b = x$

$\therefore$ $x \cdot 1 = x = 1 \cdot x$.

Thus G possesses identity element $1 \in G$.

(*iv*) **Existence of Inverse.**

For each $x = 2^a 3^b \in G$ there is $y = 2^{-a} 3^{-b} \in G$ such that $xy = 1 = yx$

[$\because$ *$a, b \in Z \Rightarrow -a, -b \in Z$*]

Thus G possesses $2^{-a} 3^{-b} \in Z$ as the inverse of $2^a 3^b \in G$.

Hence $\langle G, \cdot \rangle$ is a group.

Example 18. *Let R_2 be a set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, d are real and $ad - bc \neq 0$. Prove that R_2 is a group under matrix multiplication.*

Sol. (*i*) **Closure Axiom.**

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Thus $A, B \in R_2$.

Now $AB = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \in R_2$ [$\because$ *$|A| \neq 0, |B| \neq 0, |AB| \neq 0$*]

(*ii*) **Associative Law.**

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$ [$\because$ *Matrix multiplication is associative*]

(*iii*) **Existence of Identity.**

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as multiplicative identity because

$A \times I = A = I \times A$.

(*iv*) **Existence of Inverse.**
A^{-1} exists because $|A| \neq 0$ and belongs to R_2.
Hence R_2 is a group under multiplication.

Example 19. (*a*) *Show that the set of all matrices* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, *a and b being non-zero reals, is a group under matrix multiplication.* (P.U. 1992)

(*b*) *Show that the set of all matrices* $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, *where a, b and c are non-zero reals, is a group under matrix multiplication.* (P.U. 1992 S)

Sol. (*a*) (*i*) **Closure Axiom.**

Let $A = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$, where $a_1, b_1; a_2, b_2$ are non-zero real.

Thus $A, B \in R_2$.

Now $AB = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & b_1 b_2 \end{bmatrix} \in R_2$.

(*ii*) **Associative Law.** $(A . B). C = A . (B . C)$

[$\because$ *Matrix multiplication is associative*]

(*iii*) **Existence of Identity.**

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as multiplicative identity because

$A \times I = A = I \times A$.

(*iv*) **Existence of Inverse.**
A^{-1} exists because $|A| \neq 0$ and belongs to R_2.
Hence R_2 *i.e.*, the set of matrices of given type is a group under multiplication.

(*b*) (*i*) **Closure Axiom.**

Let $A = \begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}$, where $a_1, b_1, c_1; a_2, b_2, c_2$ are non-zero real.

Thus $A, B \in R_2$.

Now $AB = \begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ b_1 a_2 + c_1 b_2 & c_1 c_2 \end{bmatrix} \in R_2$.

(*ii*) **Associative Law.**

$(A . B) . C = A . (B . C)$ [$\because$ *Matrix multiplication is associative*]

(*iii*) **Existence of Identity.**

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as mutliplicative identity because $A \times I = A = I \times A$.

(*iv*) **Existence of Inverse.**

A^{-1} exists because $|A| = a_1 c_1 \neq 0$ and $\in R_2$.

Hence R_2 *i.e.*, the set of matrices of given type is a group under multiplication.

Example 20. *Prove that the set of matrices of the type* $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, *where* $\alpha \in R$, *is a group under matrix multiplication.*

Sol. Let $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \in G$ and $A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \in G$, where $\alpha, \beta \in R$.

(*i*) **Closure Axiom.**

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$= A_{\alpha+\beta} \in G$ $\qquad$ [$\because$ $\alpha, \beta \in R \Rightarrow \alpha+\beta \in G$]

(*ii*) **Associative Law.**

Let A_α, A_β, $A_\gamma \in G$, where $\alpha, \beta, \gamma \in R$.

Then $(A_\alpha A_\beta) A_\gamma = A_{\alpha+\beta} A_\gamma = \alpha_{(\alpha+\beta)+\gamma}$

and $A_\alpha (A_\beta A_\gamma) = A_\alpha A_{\beta+\gamma} = A_{\alpha+(\beta+\gamma)}$

Since $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$, $\qquad$ [$\because$ *Associative law holds in R*]

$\therefore$ $(A_\alpha A_\beta) A_\gamma = A_\alpha (A_\beta A_\gamma)$.

(*iii*) **Existence of Identity.**

If $A_\alpha \in G$, then $A_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} \in G$ such that $A_\alpha A_0 = A_\alpha = A_0 A_\alpha$

[$\because$ $A_\alpha A_0 = A_{\alpha+0} = A_\alpha$ *and* $A_0 A_\alpha = A_{0+\alpha} = A_\alpha$]

Thus A_0 is the identity element of G.

(*iv*) **Existence of Inverse.**

For each $A_\alpha \in G$, there is $A_{(-\alpha)} \in G$ such that $A_\alpha A_{-\alpha} = A_0 = A_{-\alpha} A_\alpha$

[$\because$ $A_\alpha A_{-\alpha} = A_{\alpha-\alpha} = A_0$ *and* $A_{-\alpha} A_\alpha = A_{-\alpha+\alpha} = A_0$]

$\therefore$ $A_{-\alpha}$ is the inverse of A_α.

Hence G is a group under matrix multiplication.

Example 21. *Show that the set G of all real valued continuous functions defined on [0, 1] is an abelian group under addition defined as* $(f+g)(x) = f(x) + g(x)$ $\forall f, g \in G$.

Sol. Let $G = \{ f \mid f(x) \in R \text{ for } x \in [0, 1] \}$, where R is the set of real valued continuous functions.

(*i*) **Closure Axiom.** Let $f, g \in G$.

Since the sum of two real valued continuous functions is a real valued continuous function,

$\therefore$ $(f+g)(x) = f(x) + g(x) \Rightarrow f + g \in G$.

Thus G is closed under addition.

(*ii*) **Associative Law.** Let $f, g, h \in G$.

Then $[(f+g)+h](x) = (f+g)(x) + h(x) = [f(x) + g(x)] + h(x)$

$= f(x) + [g(x) + h(x)]$ [$\because$ *Associative law holds in reals*]

$= [f + (g + h)](x)$

Thus $(f + g) + h = f + (g + h) \ \forall f, g, h \in G.$

(*iii*) **Existence of Identity.**

For each $f \in G$, there is $O \in G$ defined as $O(x) = 0 \ \ \forall x \in [0, 1]$

such that $(O + f)(x) = f(x) = (f + O)(x)$

Now $(O + f)(x) = O(x) + f(x) = O + f(x) = f(x) \ \forall x \in [0, 1]$

$\Rightarrow$ $O + f = f.$

Similarly $f + O = f.$

Thus $O + f = f = f + O.$

Thus $O \in G$ is the identity element.

(*iv*) **Existence of Inverse.** For each $f \in G$, there is $-f \in G$ such that $(-f)(x) = -f(x)$.

Now $[f + (-f)](x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = O(x) \ \forall x \subset [0, 1]$

$\therefore$ $f + (-f) = O.$

Similarly $(-f) + f = O.$

Thus $f + (-f) = O = (-f) + f.$

Thus each f has its additive inverse $(-f)$.

$\therefore$ G is a group under addition.

(*v*) **Commutative Law .**

$\forall f, g \in G, (f + g)(x) = f(x) + g(x)$

$= g(x) + f(x)$

$= (g + f)(x) \ \forall x \in [0, 1].$

Thus $f + g = g + f.$

Hence G is an abelian group under addition.

(*c*) **1. Ring.**

A system $\langle R, +, . \rangle$, where R is a non-empty set and addition (+), multiplication (.) are two binary compositions on R, is called a ring if it satisfies the following postulates :

Under Addition :

(*i*) **Closure Axiom.** $\forall a, b \in R \ \Rightarrow \ a + b \in R.$

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c \ \forall a, b, c \in R.$

(*iii*) **Existence of Identity.** *There exists an element $0 \in R$, called the* **identity under addition,** such that $a + 0 = a = 0 + a \ \forall a \in R.$ [0 is also called the zero-element]

(*iv*) **Existence of Inverse.** $\forall a \in R$, *there exists an element $a \in R$, called* **the inverse of a under addition,** *such that*

$a + (-a) = 0 = (-a) + a.$

(*v*) **Commutative Law.** $\forall a, b \in R, a + b = b + a.$

Under Multiplication :

(*vi*) **Closure Axiom.** $\forall a, b \in R \ \Rightarrow \ a . b \in R.$

(*vii*) **Associative Law.** $a.(b.c) = (a.b).c \ \forall a, b, c \in R.$

(*viii*) **Distributive Laws.** $\forall a, b, c \in R,$

(*i*) $a.(b + c) = a.b + a.c$ **[Left]**

(*ii*) $(b + c).a = b.a + c.a$ **[Right]**

Conclusion. (*i*) **R forms an abelian group under addition**

(*ii*) **R is a semi-group under multiplication**

(*iii*) **R satisfies distributive Laws.**

Note. The ring is called **non-associative** if associative law under multiplication does not hold.

2. Commutative Ring or Abelian Ring.

In addition to the above eight postulates, if the following postulate is also satisfied, then the ring R is called a Commutative or an Abelian Ring.

(*ix*) **Commutative Law.** $\forall\, a, b \in \text{R}, a \,.\, b = b \,.\, a$

3. Ring with Unity.

A ring R which contains the multiplicative identity (called **unity***) is called a ring with unity.* (Pbi. U. 1997)

Thus if $1 \in \text{R}$ such that $a \,.\, 1 = a = 1 \,.\, a \;\forall\, a \in \text{R}$, then the ring is called a **ring with unity.**

4. Ring without Unity.

A ring R which does not contain multiplicative identity is called a **ring without unity.**

5. Finite and Infinite Ring.

If the number of elements in the ring R is finite, then <R, +, . > is called a **finite ring**, *otherwise it is called an infinite ring.*

6. Order of Ring .

The number of elements in a finite ring is called the order of the ring.

This is denoted by O(R) or | R |.

7. Units of a ring with unity.

The elements which possess inverses under the second operation (.) are called units of a ring.

In the set I of integers, we know that $(-1)(-1) = 1$.

Thus -1 is the **unit** but not **unity** of the ring.

Again $1 \,.\, 1 = 1$.

Thus 1 is the **unit** as well as **unity** of the ring.

Note. Unity is a unit but every unit is not a unity.

8. Zero divisors of a ring.

Let < R, + , . > be a ring.

$\forall\, a, b \in \text{R}$, where $a \neq 0, b \neq 0$.

If $ab = 0$, then R is called a **ring with zero divisors.**

Here a is called the **left-zero divisor** and b the **right-zero divisor.**

An element which is left as well as right-zero divisors is called the **zero divisor of the ring.**

In abelian rings, every left-zero divisor is also the right-zero divisor and vice-versa.

In non-abelian rings, there may be some elements which are simply left-zero divisor or simply right-zero divisors.

9. Ring without zero-divisor.

The ring which is not with zero divisor is called the ring without zero divisor

i.e., if $a \neq 0, b \neq 0$, then $ab \neq 0$.

SOLVED EXAMPLES

Example 1. *Prove that <Z, + ,.>, where Z is a set of all integers, is a ring.* (G.N.D.U. 1998)

Sol. The system is < Z, +, .>, where

$$Z = \{\ldots\ldots\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$$

and '+' and '.' are binary compositions in Z.

Under Addition :

(*i*) **Closure Axiom.** $\forall\, a, b \in Z \Rightarrow a + b \in Z$. [$\because$ *Sum of two integers is an integer*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c\ \forall\, a, b, c \in Z$.

(*iii*) **Existence of Identity.** There exists an element $0 \in Z$ such that $a + 0 = a = 0 + a\ \forall\, a \in Z$.

(*iv*) **Existence of Inverse.** $\forall\, a \in Z$, there exists an element $-a \in Z$ such that $a + (-a) = 0 = (-a) + a$.

(*v*) **Commutative Law.** $\forall\, a, b \in Z, a + b = b + a$.

Under Multiplication :

(*vi*) **Closure Axiom.** $\forall\, a, b \in Z \Rightarrow a \,.\, b \in Z$. [$\because$ *Product of two integers is an integer*]

(*vii*) **Associative Law.** $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c\ \forall\, a, b, c \in Z$.

(*viii*) **Distributive Laws.** $\forall\, a, b, c \in Z$,

(I) $a \,.\, (b + c) = a \,.\, b + a \,.\, c$

(II) $(b + c) \,.\, a = b \,.\, a + c \,.\, a$

Hence $< Z, +, . >$ is a ring.

Example 2. (*a*) *Is the set of even integers a ring under usual addition and multiplication ? Does it contain identity ?* (Pbi. U. 1997)

(*b*) *Prove that $<M, +, \cdot>$, where M is a set of those integers which are multiples of 5, is a ring.*

Sol. (*a*) Let the system be $< E, +, \cdot >$, where

$E = \{\ldots\ldots\ldots -4, -2, 0, 2, 4, \ldots\ldots\}$ and '+' and '.' are binary composition in E.

Under Addition :

(*i*) **Closure Axiom.** $\forall\, a, b \in E \Rightarrow a + b \in E$ [$\because$ *Sum of two even integers is an even integer*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c\ \forall\, a, b, c \in E$.

(*iii*) **Existence of Identity.** There exists an element $0 \in E$ such that $a + 0 = a = 0 + a\ \forall\, a \in E$.

(*iv*) **Existence of Inverse.** $\forall\, a \in E$, there exists an element $-a \in E$ such that

$$a + (-a) = 0 = (-a) + a.$$

(*v*) **Commutative Law.** $\forall\, a, b \in E \Rightarrow a + b = b + a$.

Under Multiplication :

(*vi*) **Closure Axiom.** $\forall\, a, b \in E \Rightarrow a \,.\, b \in E$ [$\because$ *Product of two even integers is an even integer*]

(*vii*) **Associative Law.** $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c\ \forall\, a, b, c \in E$.

(*viii*) **Distributive laws.** $\forall\, a, b, c \in E$,

(I) $a \,.\, (b + c) = a \,.\, b + a \,.\, c$

(II) $(b + c) \,.\, a = b \,.\, a + c \,.\, a$

Hence $<E, +, \cdot>$ is a ring and it contains identity.

(*b*) The system is $< M, +, \cdot >$, where

$M = \{\ldots\ldots\ldots, -10, -5, 0, 5, 10, \ldots\ldots\}$

and '+' and '.' are binary compositions in M.

Under Addition :

(*i*) **Closure Axiom.** Let $a = 5k_1$ and $b = 5k_2$, where $k_1, k_2 \in Z$.

Then $a + b = 5k_1 + 5k_2 = 5\,(k_1 + k_2) \in M$.

(*ii*) **Associative Law.** Let $a = 5k_1, b = 5k_2, c = 5k_3$; where $k_1, k_2, k_3 \in Z$.

Then $a + (b + c) = 5k_1 + (5k_2 + 5k_3) = (5k_1 + 5k_2) + 5k_3 = (a + b) + c$.

(*iii*) **Existence of Identity.** There exists an element $0 \in M$ such that $a + 0 = a = 0 + a \; \forall \; a \in M$.

Let $a = 5k_1$.

Then $a + 0 = 5k_1 + 0 = 5k_1 = a$ and $0 + a = 0 + 5k_1 = 5k_1 = a$

$\therefore$ 0 is the identity element of M.

(*iv*) **Existence of Inverse.** $\forall \; a \in M$, there exists an element $-a \in M$ such that $a + (-a) = 0 = (-a) + a$.

(*v*) **Commutative Law.** $\forall \; a, b \in M \Rightarrow a + b = b + a$.

Under Multiplication :

(*vi*) **Closure Axiom.** $a \,.\, b = (5k_1)(5k_2) = 25k_1 k_2 \in M$.

(*vii*) **Associative Law.** $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c \; \forall \; a, b, c \in M$.

(*viii*) **Distributive Laws.** $\forall \; a, b, c \in M$,

(I) $a \,.\, (b + c) = a \,.\, b + a \,.\, c$ (II) $(b + c) \,.\, a = b \,.\, a + c \,.\, a$.

Hence $< M, +, \cdot >$ is a ring.

Example 3. *Prove that* $< Q, +, \cdot >$, *where* Q *is a set of all rational numbers, is a ring.*

Sol. The system is $<Q, +, \cdot>$, where Q is a set of rational numbers and '+' and '.' are binary compositions in Q.

Under Addition :

(*i*) **Closure Axiom.** $\forall \; a, b \in Q \Rightarrow a + b \in Q$.

[$\because$ *Sum of two rational numbers is a rational number*]

(*ii*) **Associative Law.** $a + (b + c) = (a + b) + c \; \forall \; a, b, c \in Q$.

(*iii*) **Existence of Identity.** There exists an element $0 \in Q$ such that

$a + 0 = a = 0 + a \; \forall \; a \in Q$.

(*iv*) **Existence of Inverse.** $\forall \; a \in Q$, there exists an element $-a \in Q$ such that $a + (-a) = 0 = (-a) + a$.

(*v*) **Commutative Law.** $\forall \; a, b \in Q, a + b = b + a$.

Under Multiplication :

(*vi*) **Closure Axiom.** $\forall \; a, b \in Q \Rightarrow a \,.\, b \in Q$.

[$\because$ *Product of two rational numbers is a rational number*]

(*vii*) **Associative Law.** $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c \; \forall \; a, b, c \in Q$.

(*viii*) **Distributive Laws.** $\forall \; a, b, c \in Q$,

(I) $a \,.\, (b + c) = a \,.\, b + a \,.\, c$ (II) $(b + c) \,.\, a = b \,.\, a + c \,.\, a$

Hence $<Q, +, \cdot>$ is a ring.

Example 4. *Prove that* $<R, +, \cdot>$, *where* R *is a set of all real numbers, is a ring.*

Sol. Exactly similar to Ex. 3. [*Replace* Q *by* R]

Example 5. *Prove that the set* R_2 *of all ordered pairs* (a, b) *of real numbers is a commutative ring under the addition and multiplication of ordered pairs defined as*

$(a, b) + (c, d) = (a + c, b + d)$

$(a, b)(c, d) = (ac, bd) \; \forall \; (a, b), (c, d) \in R$. (P.U. 1992)

Sol. The system is $< S, +, \cdot >$, where $S = \{ (x, y) \mid x, y \in R \}$

Under Addition :

(*i*) **Closure Axiom.** $\forall \; (a, b), (c, d) \in S \Rightarrow (a, b) + (c, d) \in S$

[$\because$ $(a, b) + (c, d) = (a + c, b + d) \in S$ *as* $a, c \in R \Rightarrow a + c \in R$ *and* $b, d \in R \Rightarrow b + d \in R$]

(*ii*) **Associative Law.** $\forall\, (a, b), (c, d), (e, f) \in S$

$$(a, b) + ((c, d) + (e, f)) = ((a + b) + (c, d)) + (e, f)$$

$$[\because\ (a, b) + ((c, d) + (e, f)) = (a, b) + (c + e, d + f) = (a + (c + e), b + (d + f))$$
$$= ((a + c) + e, (b + d) + f) = ((a, b) + (c, d)) + (e, f)]$$

(*iii*) **Existence of Identity.** There exists an element $(0, 0) \in S$ such that $(x, y) + (0, 0) = (x, y) = (0, 0) + (x, y)\ \forall\ (x, y) \in S$.

(*iv*) **Existence of Inverse.** $\forall\ (x, y) \in S$, there exists an element $(-x, -y) \in S$, such that $(x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y)$.

(*v*) **Commutative Law.** $\forall\, (a, b), (c, d) \in Z$

$(a, b) + (c, d) = (c, d) + (a, b)$. $\quad [\because\ (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d) + (a, b)]$

Under Multiplication :

(*vi*) **Closure Axiom.** $\forall\, (a, b), (c, d) \in S \Rightarrow (a, b)(c, d) \in S$

$$[\because\ (a, b)(c, d) = (ac, bd) \in Z \text{ as } a, c \in R \Rightarrow ac \in R \text{ and } b, d \in R \Rightarrow bd \in R]$$

(*vii*) **Associative Law.** $\forall\, (a, b), (c, d), (e, f) \in S$

$$(a, b) . ((c, d) . (e, f)) = ((a, b) . (c, d)) . (e, f)$$

$$[\because\ (a, b) . ((c, d) . (e, f)) = (a, b) . (ce, df) = (a(ce), b . (df)) = ((ac)e, (bd)f)$$
$$= (ac, bd)(e, f) = ((a, b) . (c, d)) . (e, f)]$$

(*viii*) **Distributive Law.** $\forall\, (a, b), (c, d), (e, f) \in S$,

(I) $(a, b) . ((c, d) + (e, f)) = (a, b) . (c, d) + (a, b) . (e, f)$

(II) $((c, d) + (e, f)) . (a, b) = (c, d) . (a, b) + (e, f) . (a, b)$

Hence $< S, +, . >$ is a commutative Ring.

Example 6. *Prove that the set of matrices of order 2 × 2 forms a ring w.r.t. addition and multiplication of matrices.* (G.N.D.U. 1992)

Sol. Let S be the set of given matrices.

Let A, B, C be any three elements of S such that

$$A = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix}, B = \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix}, C = \begin{bmatrix} 0 & a_3 \\ 0 & b_3 \end{bmatrix},$$

where a_1, b_1 ; a_2, b_2 ; a_3, b_3 are real numbers.

Under Addition :

(*i*) **Closure Axiom.**

$$A + B = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} 0 & a_1 + a_2 \\ 0 & b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 0 & a' \\ 0 & b' \end{bmatrix}, \text{ where } a_1 + a_2 = a' \text{ and } b_1 + b_2 = b'.$$

Since $a_1 + a_2 = a'$ and $b_1 + b_2 = b'$ also belong to R ,

$\therefore\quad \begin{bmatrix} 0 & a' \\ 0 & b' \end{bmatrix} \in S.$

$\therefore$ S is closed w.r.t. addition.

(*ii*) **Associative Law.**

We know that addition of matrices is associative. [*Verify* !]

(*iii*) **Existence of Identity.**

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero element of S.

(*iv*) **Existence of Inverse.**

Additive inverse of $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in$ S is $\begin{bmatrix} 0 & -a \\ 0 & -b \end{bmatrix}$ because $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & -a \\ 0 & -b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Obviously $\begin{bmatrix} 0 & -a \\ 0 & -b \end{bmatrix} \in$ S as $-a, -b \in$ R.

Thus additive inverse of every element of S exists in S.

(*v*) **Commutative Law.**

We know that addition of matrices is commutative. [*Verify* !]

Under Multiplication :

(*vi*) **Closure Axiom.**

$$A.B = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} . \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix}$$

Since a_1b_2 and $b_1b_2 \in$ R, $\therefore$ A.B $\in$ S

$\therefore$ S is closed w.r.t. multiplication.

(*vii*) **Associative Law.**

We know that multiplication of matrices is associative. [*Verify* !]

(*viii*) **Distributive Law.**

We know that multiplication of matrices is distributive w.r.t. addition. [*Verify* !]

Hence the given system is a ring.

Example 7. *Prove that the set M of all n $\times$ n matrices over reals is a non-commutative ring with unity, with zero divisors under addition and multiplication of matrices.*

Sol. Let A, B, C $\in$ M be $n \times n$ matrices over reals

$\Rightarrow$ $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$, $C = [c_{ij}]_{n \times n}$, where $a_{ij}, b_{ij}, c_{ij} \in$ R for $1 \le i \le n$, $1 \le j \le n$.

(*a*) **Under Addition :**

(*i*) **Closure Axiom.** $\forall$ A, B $\in$ M, $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{n \times n} \in M$

$[\because \; a_{ij}, b_{ij} \in R \Rightarrow a_{ij} + b_{ij} \in R]$

(*ii*) **Associative Law.** $\forall$ A, B, C $\in$ M,

$$\begin{aligned} A + (B + C) &= [a_{ij}]_{n \times n} + \left([b_{ij}]_{n \times n} + [c_{ij}]_{n \times n} \right) \\ &= [a_{ij}]_{n \times n} + [b_{ij} + c_{ij}]_{n \times n} \\ &= [a_{ij} + (b_{ij} + c_{ij})]_{n \times n} \\ &= [(a_{ij} + b_{ij}) + c_{ij}]_{n \times n} \qquad [\because \textit{ Associative Law holds in } R] \\ &= [a_{ij} + b_{ij}]_{n \times n} + [c_{ij}]_{n \times n} = (A + B) + C. \end{aligned}$$

(*iii*) **Existence of Identity.**

For A $\in$ M, there exists O = $[0]_{n \times n} \in$ M such that A + O = A = O + A.

Now A + O = $[a_{ij}]_{n \times n} + [0]_{n \times n} = [a_{ij} + 0]_{n \times n} = [a_{ij}]_{n \times n}$ = A.

Similarly O + A = A.

$\therefore$ A + O = A = A + O.

Thus O is the additive identity.

(*iv*) **Existence of Inverse.**

For A $\in$ M, there exists – A $\in$ M such that A + (– A) = O = (– A) + A.

Now A + (– A) $= [a_{ij}]_{n \times n} + [-a_{ij}]_{n \times n}$ $\qquad [\because -A = [-a_{ij}]_{n \times n}]$

$= [a_{ij} + (-a_{ij})]_{n \times n} = [0]_{n \times n}$ = O.

Similarly (– A) + A = O.

$\therefore$ A + (– A) = O = (– A) + A.

$\therefore$ – A is the inverse of A $\in$ M and – A $\in$ M.

Under Multiplication :

(*vi*) **Closure Axiom.**

Let $\quad A = [a_{ij}]_{n \times n}$ and $B = [b_{jk}]_{n \times n} \in$ M.

Then $AB = [c_{ik}]_{n \times n}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} \in$ R

$\Rightarrow$ AB $\in$ M.

(*vii*) **Associative Law.**

Let $\quad A = [a_{ij}]_{n \times n}$, $B = [b_{jk}]_{n \times n}$ and $C = [c_{kp}]_{n \times n} \in$ M.

Then $AB = [d_{ik}]_{n \times n}$, where $d_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$

and $\qquad BC = [e_{jp}]_{n \times n}$, where $e_{jp} = \sum_{k=1}^{n} b_{jk}\; c_{kp}$

Now (i, p)th element of A (BC)

= (ith row of A) (pth column of BC)

$$= \sum_{j=1}^{n} a_{ij}\, e_{jp} = \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{n} b_{jk}\, c_{kp} \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij}\, b_{jk}\, c_{kp} = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kp}$$

Similarly (i, p)th element of (AB) C = $\sum_{k=1}^{n} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kp}$

$\therefore$ A (BC) = (AB) C.

(*viii*) **Distributive Law.**

Let $\qquad A = [a_{ij}]_{n \times n}$, $B = [b_{jk}]_{n \times n}$ and $C = [c_{jk}]_{n \times n} \in$ M.

Then $\quad B + C = [\, b_{jk} + c_{jk}\,]_{n \times n}$

$\therefore$ (i, k)th element of $A(B+C) = \sum_{k=1}^{n} a_{ij}(b_{jk} + c_{jk})$

$$= \sum_{k=1}^{n}(a_{ij}\, b_{jk} + a_{ij} c_{jk}) = \sum_{k=1}^{n} a_{ij} \,.\, b_{jk} + \sum_{k=1}^{n} a_{ij} \,.\, c_{jk}$$

$= (i, k)$th element of AB + (i, k)th element of AC

$= (i, k)$th element of (AB + AC)

Also A(B + C) = AB + AC are of type $n \times n$.

Thus *M is a ring*

(*b*) In general, matrix multiplication is not commutative,

$\therefore$ M is non-commutative ring.

(*c*) Since $I_{n \times n} \in M$ such that AI = A = IA $\forall$ A $\in$ M.

Thus I is a multiplicative identity.

(*d*) We know that if A $\neq$ O, B $\neq$ O, but AB = O. [*Verify* !]

Hence M is non-commutative ring, with unity and with zero divisors.

Example 8. *Give an example of the following :*

(i) A commutative ring without unity

(ii) A non-commutative ring with unity

(iii) A ring with zero divisors

(iv) A non-commutative ring.

Sol. (*i*) The ring of even integers is a commutative ring without unity.

(*ii*) The ring of 2×2 matrices over reals is a non-commutative ring with unity.

(*iii*) The ring of 2×2 matrices is a ring such that [A] $\neq$ O, [B] $\neq$ O still AB = O.

(*iv*) The ring of matrices is non-commutative.

Example 9. (*i*) *If $a^2 = a$ $\forall a \in R$, then $a + a = 0$.*

(ii) If every $x \in R$ satisfies $x^2 = x$, prove that R must be commutative. (G.N.D.U. 1981)

Sol. (*i*) $(a + a)^2 = (a + a)$ [*Given*]

$\Rightarrow$ $(a + a)(a + a) = a + a$

$\Rightarrow$ $a^2 + a^2 + a^2 + a^2 = a + a$ [*Distributive Laws*]

$\Rightarrow$ $a + a + a + a = a + a$

$\Rightarrow$ $a + a = 0.$ [*Cancellation Laws*]

(*ii*) Let $a, b \in R \Rightarrow a + b \in R$

We have $(a + b)^2 = a + b$

$\Rightarrow$ $a^2 + ab + ba + b^2 = a + b$

$\Rightarrow$ $a + ab + ba + b = a + b$ [$\because$ $a^2 = a$ *and* $b^2 = b$]

$\Rightarrow$ $(a + b) + (ab + ba) = a + b$

$\Rightarrow$ $ab + ba = 0$ [*Cancellation Laws*]

$\Rightarrow$ $ab + ba = ab + ab$

$\Rightarrow$ $ba = ab$ [*Left Cancellation Law*]

$\Rightarrow$ R is commutative.

Example 10. *If R is a system satisfying all the conditions for a ring with unit element with the possible exception $a + b = b + a$, prove that R is a ring.*

Sol. $(a+b).(1+1) = (a+b).1 + (a+b).1 = a.1 + b.1 + a.1 + b.1$
$= a + (b+a) + b$

Again $(a+b).(1+1) = a.(1+1) + b.(1+1) = a.1 + a.1 + b.1 + b.1$
$= a + (a+b) + b$

Then $a + (b+a) + b = a + (a+b) + b$

$\Rightarrow$ $b + a = a + b$

$\Rightarrow$ addition is commutative.

Hence R is a ring.

Example 11. *Prove that the set $G = \{a + \sqrt{2}\, b \mid a, b \in Q\}$, where Q is the set of rationals, is a ring.*

Sol. We have $G = \{a + \sqrt{2}\, b \mid a, b \in Q\}$, where Q is the set of rationals.

Under Addition :

(*i*) **Closure Axiom.**

Let $x = l + \sqrt{2}\, m$ and $y = n + \sqrt{2}\, p$, where $l, m, n, p \in Q$.

Now $x + y = (l + \sqrt{2}\, m) + (n + \sqrt{2}\, p) = (l+n) + \sqrt{2}\,(m+p) \in G$.

[$\because$ *$l, n \in Q \Rightarrow l + n \in Q$ and $m, p \in Q \Rightarrow m + p \in Q$*]

(*ii*) **Associative Law.**

Let $x = l + \sqrt{2}\, m$, $y = n + \sqrt{2}\, p$ and $z = q + \sqrt{2}\, r$, where $l, m, n, p, q, r \in Q$.

Now $x + (y+z) = (l + \sqrt{2}\, m) + [(n + \sqrt{2}\, p) + (q + \sqrt{2}\, r)] = (l + \sqrt{2}\, m) + [(n+q) + \sqrt{2}\,(p+r)]$

$= [l + (n+q)] + \sqrt{2}\,[m + (p+r)]$

$= [(l+n) + q] + \sqrt{2}\,[(m+p) + r]$ [$\because$ *Associative law holds in Q*]

$= [(l+n) + \sqrt{2}\,(m+p)] + (q + \sqrt{2}\, r)$

$= [(l + \sqrt{2}\, m) + (n + \sqrt{2}\, p)] + (q + \sqrt{2}\, r) = (x+y) + z.$

(*iii*) For $x = l + \sqrt{2}\, m \in G$, there is $0 = 0 + \sqrt{2}\, 0 \in G$ such that
$x + 0 = x = 0 + x.$ [*Verify* !]

$\therefore$ G possess $0\ (= 0 + \sqrt{2}\, 0)$ as the identity element.

(*iv*) For $x = l + \sqrt{2}\, m \in G$, there is $y = -l + \sqrt{2}\,(-m) \in G$ such that $x + y = 0 = y + x$.

Now $x + y = (l + \sqrt{2}\, m) + ((-l) + \sqrt{2}\,(-m)) = (l + (-l)) + \sqrt{2}\,(m + (-m)) = 0 + \sqrt{2}\,(0) = 0$

Similarly $y + x = 0$.

$\therefore$ $x + y = 0 = y + x.$

Thus y is the inverse of x.

Hence <G, +> is a group.

(*v*) **Commutative Law.**

Let $x = l + \sqrt{2}\, m$ and $y = n + \sqrt{2}\, p \in G$.

$\therefore$ $x + y = y + x$

$\therefore \quad x+y = (l+\sqrt{2}\,m)+(n+\sqrt{2}\,p) = (l+n)+\sqrt{2}\,(m+p)$

$= (n+l)+\sqrt{2}\,(p+m)$ [$\because$ *Commutative Law holds in Q*]

$= (n+\sqrt{2}\,p)+(l+\sqrt{2}\,m) = y+x.$

Hence <G, +> is an abelian group.

Under Multiplication :

(*vi*) **Closure Axiom.**

Let $x = l+\sqrt{2}\,m$ and $y = n+\sqrt{2}\,p$, where $l, m, n, p \in Q$.

$\therefore \quad xy = (l+\sqrt{2}\,m)\,(n+\sqrt{2}\,p) = (ln+2\,mp)+\sqrt{2}\,(lp+mn) \in G$

[$\because$ $l, m, n, p \in Q \Rightarrow ln+2\,mp,\ lp+mn \in Q$]

(*vii*) **Associative Law.**

Let $x = l+\sqrt{2}\,m,\ y = n+\sqrt{2}\,p,\ z = q+\sqrt{2}\,r \in G$, where $l, m, n, p, q, r \in Q$.

Now $x(yz) = (l+\sqrt{2}\,m)\left[(n+\sqrt{2}\,p)\,(q+\sqrt{2}\,r)\right] = (l+\sqrt{2}\,m)\left[(nq+2\,pr)+\sqrt{2}\,(pq+nr)\right]$

$= l\,(nq+2\,pr)+2\,m\,(pq+nr)+\sqrt{2}\,[\,l\,(pq+nr)+m\,(nq+2\,pr)]$

$= (l\,nq+2\,lpr+2\,mpq+2\,mnr)+\sqrt{2}\,(lpq+lnr+mnq+2\,mpr)$

Similarly $(xy)z = (lnq+2\,lpr+2\,mpq+2\,mnr)+\sqrt{2}\,(lpq+lnr+mnq+2\,mpr)$

$\therefore \quad x\,(y\,z) = (x\,y)\,z.$

(*viii*) **Distributive Laws.**

Let $x = l+\sqrt{2}\,m,\ y = n+\sqrt{2}\,p,\ z = q+\sqrt{2}\,r$, where $l, m, n, p, q, r \in Q$

$\therefore \quad x\,.\,(y+z) = (l+\sqrt{2}\,m)\,.\left[(n+\sqrt{2}\,p)+(q+\sqrt{2}\,r)\right] = (l+\sqrt{2}\,m)\,.\left[(n+q)+\sqrt{2}\,(p+r)\right]$

$= \left[l\,(n+q)+2\,m\,(p+r)\right]+\sqrt{2}\,\left[l\,(p+r)+m\,(n+q)\right]$

$= (ln+lq+2\,mp+2\,mr)+\sqrt{2}\,(lp+lr+mn+mq)$

$= \left[(ln+2\,mp)+(lq+2\,mr)\right]+\sqrt{2}\,\left[(lp+mn)+(lr+mq)\right]$

and $\quad x\,y+x\,z = (l+\sqrt{2}\,m)\,(n+\sqrt{2}\,p)+(l+\sqrt{2}\,m)\,(q+\sqrt{2}\,r)$

$= \left[(ln+2\,mp)+\sqrt{2}\,(lp+mn)\right]+\left[(lq+2mr)+\sqrt{2}\,(lr+mq)\right]$

$= \left[(ln+2\,mp)+(lq+2\,mr)\right]+\sqrt{2}\,\left[(lp+mn)+(lr+mq)\right]$

$\therefore \quad x\,.\,(y+z) = x\,y+x\,z.$

Similarly $(y+z)\,.\,x = y.\,x+z\,x.$

Hence < G, +, . > is a ring.

Example 12. *Prove that Q is a ring under the compositions* $\oplus$ and $\odot$ *defined as* $a \oplus b = a+b-1$ *and* $a \odot b = a+b-a\,b$, *where* $a, b \in Q$ *and Q is a set of rational numbers.*

Sol. Under Addition :

(*i*) **Closure Axiom.**

Let $a, b \in Q \Rightarrow a+b-1 \in Q$ [$\because$ $a, b \in Q \Rightarrow a+b \in Q \Rightarrow a+b-1 \in Q$]

$\Rightarrow a \oplus b \in Q.$

(*ii*) **Associative Law.**

Let $a, b, c \in Q$.

Then $a \oplus (b \oplus c) = a \oplus (b + c - 1) = a + (b + c - 1) - 1 = a + b + c - 2$

and $(a \oplus b) \oplus c = (a + b - 1) \oplus c = (a + b - 1) + c - 1 = a + b + c - 2$

$\therefore \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c.$

(*iii*) **Existence of Identity.**

For each $a \in Q$, there exists $e \in Q$ such that $a \oplus e = a = e \oplus a$.

Now $a \oplus e = a + e - 1$

$\therefore \quad a \oplus e = a \Rightarrow a + e - 1 = a \Rightarrow e = 1 \in Q.$

Similarly $e \oplus a = a \Rightarrow e = 1 \in Q$.

(*iv*) **Existence of Inverse.**

For each $a \in Q$, there exists $b \in Q$ such that $a \oplus b = e = b \oplus a$.

Now $a \oplus b = a + b - 1 = 1 \Rightarrow b = 2 - a \in Q$

Similarly $b \oplus a = e = 1 \Rightarrow b = 2 - a \in Q$.

Thus $b\ (= 2 - a)$ is the inverse of $a \in Q$.

(*v*) **Commutative Law.**

$\forall\, a, b \in Q, a \oplus b = a + b - 1 \in Q \Rightarrow b + a - 1 \in Q \Rightarrow b \oplus a.$

Under Multiplication :

(*vi*) **Closure Axiom.**

Let $a, b \in Q \Rightarrow a + b - ab \in Q$ $\quad [\because a, b \in Q \Rightarrow a + b, ab \in Q \Rightarrow a + b - ab \in Q]$

$\Rightarrow a \odot b \in Q.$

(*vii*) **Associative Law.**

Let $a, b, c \in Q$.

Then $a \odot (b \odot c) = a \odot (b + c - bc) = a + (b + c - bc) - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc.$

Similarly $(a \odot b) \odot c = a + b + c - bc - ab - ac + abc.$

$\therefore \quad a \odot (b \odot c) = (a \odot b) \odot c.$

(*viii*) **Distributive Laws.**

Let $a, b, c \in Q$.

Then $a \odot (b \oplus c) = a \odot (b + c - 1) = a + (b + c - 1) - a(b + c - 1)$

$= a + b + c - 1 - ab - ac + a = 2a + b + c - ab - ac - 1$

and $(a \odot b) \oplus (a \odot c) = (a + b - ab) \oplus (a + c - ab) = (a + b - ab) + (a + c - ac) - 1$

$= 2a + b + c - ab - ac - 1$

$\therefore \quad a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).$

Similarly $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)\ \forall\, a, b, c \in Q.$

Hence Q is a ring under given operation.

Addition and Multiplication Modulo *m*

(*i*) **Addition modulo *m*.**

If a and b are two integers, then by **addition modulo *m*** expressed as $a +_m b$ we mean least non-negative number r which is the remainder when $a + b$ is divided by m.

Thus $a \underset{m}{+} b = r$, where $0 \leq r < m$.

For Examples : $3 \underset{6}{+} 20 = 23 = 6\,(3) + 5 = 5$

$7 \underset{6}{+} 5 = 12 = 6(2) + 0 = 0.$

(*ii*) **Multiplication modulo *m***

If a and b are two integers, then by **multiplication modulo *m*** expressed as $a \underset{m}{\times} b$ we mean least non-negative number r which is the remainder, where $a\,b$ is divided by m.

Thus $a \underset{m}{\times} b = r$, where $0 \leq r < m$.

For Examples : $3 \underset{7}{\times} 20 = 60 = 7\,(8) + 4 = 4$

$4 \underset{6}{\times} 3 = 12 = 6\,(2) + 0 = 0.$

Example 13. *Show that the set $Z_7 = \{0, 1, 2, \ldots\ldots, 6\}$ forms a ring under addition and multiplication modulo 7.* (Pbi. U. 1996)

Sol. The given set is $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

The composition tables for addition and multiplication modulo 7 are :

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	2	4	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Under Addition :

(*i*) **Closure Axiom.** $\forall\, a, b \in Z_7 \;\Rightarrow\; a + b \in Z_7$.

(*ii*) **Associative Law.** $\forall\, a, b, c \in Z_7$, $a + (b + c) = (a + b) + c$.

[$\because$ *$a + (b + c)$ and $(a + b) + c$ will have same +ve remainder when $a + b + c$ is divided by 7*]

(*iii*) **Existence of Identity.**

Here the element 0 is the identity element because $a + 0 = a = 0 + a \;\forall\, a \in Z_7$.

(*iv*) **Existence of Inverse.**

Here inverse of 4 is 3 and 3 is 4 ; and so on. [$\because$ *$4 + 3 = 0$ and $3 + 4 = 0$*]

Thus the inverse of $a \in Z_7$ is obtained by subtracting a from 7.

(*v*) **Commutative Law.**

$\forall\, a, b \in Z_7 \;\Rightarrow\; a + b = b + a \pmod 6$

Hence **Z_7 is an abelian group.**

Under Multiplication :

(*vi*) **Closure Axiom .**

$\forall\, a, b \in Z_7 \;\Rightarrow\; a\,.\,b \in Z_7$

(*vii*) **Associative Law.**

$$\forall\, a, b, c \in Z_7,\ (a \underset{7}{\times} b) \underset{7}{\times} c = a \underset{7}{\times} (b \underset{7}{\times} c)$$

(*viii*) **Distributive Laws.**

$$\forall\, a, b, c \in Z_7,\ a \underset{7}{\times} (b \underset{7}{+} c) = a \underset{7}{\times} b \underset{7}{+} a \underset{7}{\times} c$$

and $(b \underset{7}{+} c) \underset{7}{\times} a = b \underset{7}{\times} a \underset{7}{+} c \underset{7}{\times} a$.

Hence Z_7 forms a ring.

5. Field

Consider Q, the set of all rational numbers.

Q admits two operations *viz.* addition (+) and multiplication (.).

Q is an abelian group under addition.

Multiplication in Q is associative as well as commutative.

$1 \in Q$ such that $1 \,.\, x = x\ \forall\, x \in Q$.

Here 1 is called the *multiplicative identity* or *unity.*

For each non-zero $x \in Q$, there exists $x' \left(= \dfrac{1}{x}\right)$ such that $x \,.\, x' = 1$.

Thus every non-zero element of Q has inverse under multiplication.

Further for any x, y, z in Q, the following property holds :

$x \,.\, (y + z) = x \,.\, y + x \,.\, z.$ **(Distribution of multiplication over addition)**

Def. *A set F having at least two elements and two operations addition (+) and multiplication (.) are defined, which for every ordered pair (x, y) of elements of F, determine elements* $x + y$, $x\,y$ *in F is said to be a field if it satisfies the following postulates :* (Pbi. U. 1997)

(*a*) **For Addition :**

(*i*) **Commutativity.** $x + y = y + x\ \ \forall\, x, y \in F$.

(*ii*) **Associativity.** $(x + y) + z = x + (y + z)\ \ \forall\, x, y, z \in F$.

(*iii*) **Existence of Identity.** There exists $0 \in F$ (called **zero element** of F) such that $x + 0 = x = 0 + x\ \forall\, x \in F$.

(*iv*) **Existence of Inverse.** There exists $x' \in F$ (called **negative** of x) such that $x + x' = 0 = x' + x\ \ \forall\, x \in F$.

(*b*) **For Multiplication :**

(*i*) **Commutativity.** $x \,.\, y = y \,.\, x\ \ \forall\, x, y \in F$.

(*ii*) **Associativity.** $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)\ \forall\, x, y, z \in F$.

(*iii*) **Existence of Identity.** There exists $1 \in F$ (called **unity** of F) such that $x \,.\, 1 = x = 1 \,.\, x\ \forall\, x \in F$.

(*iv*) **Existence of Inverse.** There exists $x'\ (\neq 0) \in F$ (called **reciprocal** of x) such that $x \,.\, x' = 1 = x' \,.\, x\ \forall\, x \in F$.

(*c*) **Distributive Laws.** For all x, y, z in F, the following hold :

Left. $x \,.\, (y + z) = x \,.\, y + x \,.\, z$

Right. $(y + z) \,.\, x = y \,.\, x + z \,.\, x$

In above, it is observed that :

(*i*) Postulates (*a*) (*i*) — (*iv*) show that a field F is an abelian group under addition.

(*ii*) Postulates (*b*) (*i*) – (*iv*) show that a field F* (of all non-zero elements of F) is an abelian group under multiplication.

SOLVED EXAMPLES

Example 1. *Show that the set of numbers of the form $a + b\sqrt{2}$ with a, b as rationals is a field.* (G.N.D.U. 1990)

Sol. Let $Q(\sqrt{2}) = \{a + b\sqrt{2} \; ; a, b \in Q\}$.

(*a*) **Under Addition.**

(*i*) **Commutative.** $x + y = y + x \quad \forall x, y \in Q\sqrt{2}$ $\left[\because \text{ If } x = a + b\sqrt{2}, y = c + d\sqrt{2} \; \therefore \; x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (c + d\sqrt{2}) + (a + b\sqrt{2}) = y + x\right]$

(*ii*) **Associative.** $(x + y) + z = x + (y + z) \;\forall x, y, z \in Q(\sqrt{2})$.

(*iii*) **Existence of Identity.** There exists $0 \in Q(\sqrt{2})$ such that

$x + 0 = x = 0 + x \;\forall x \in Q(\sqrt{2})$ [*Here* $0 = 0 + 0\sqrt{2}$]

(*iv*) **Existence of Inverse.** There exists $x' \in Q(\sqrt{2})$ such that

$x + x' = 0 = x' + x \;\forall x \in Q(\sqrt{2})$ [*If* $x = a + b\sqrt{2}$, *then* $x' = (-a) + (-b)\sqrt{2}$]

(*b*) **Under Multiplication.**

(*i*) **Commutativity.** $x \,.\, y = y \,.\, x \;\forall x, y \in Q(\sqrt{2})$.

(*ii*) **Associative.** $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z) \;\forall x, y, z \in Q(\sqrt{2})$.

(*iii*) **Existence of Identity.** There exists $1 \in Q(\sqrt{2})$ such that

$x \,.\, 1 = x = 1 \,.\, x \;\forall x \in Q(\sqrt{2})$ [*Here* $1 = 1 + 0\sqrt{2}$]

(*iv*) **Existence of Inverse.** There exists $x' \in Q(\sqrt{2})$ such that

$x \,.\, x' = 1 = x' \,.\, x \;\forall x \in Q(\sqrt{2})$.

(*c*) **Distributive Laws.** $\forall x, y, z \in Q(\sqrt{2})$, the following hold :

Left. $x \,.\, (y + z) = x \,.\, y + x \,.\, z$

Right. $(y + z) \,.\, x = y \,.\, x + z \,.\, x$

Hence $Q(\sqrt{2})$ forms a field.

Example 2. *Prove that the set R of real numbers is a field w.r.t. the addition and multiplication compositions defined as it.* (G.N.D.U. 1991)

Sol. (*a*) **Under Addition :**

(*i*) **Commutative.** $x + y = y + x \;\forall x, y \in R$.

(*ii*) **Associative.** $(x + y) + z = x + (y + z) \;\forall x, y, z \in R$.

(*iii*) **Existence of Identity.** There exists $0 \in R$ such that $x + 0 = x = 0 + x \;\forall x \in R$.

(*iv*) **Existence of Inverse.** There exist $y \in R$, where $y = -x$ such that $x + y = 0 = y + x \;\forall x \in R$.

(*b*) **Under Multiplication :**

(*i*) **Commutative.** $x \,.\, y = y \,.\, x \;\forall x, y \in R$.

(*ii*) **Associative.** $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z) \;\forall x, y, z \in R$.

(*iii*) **Existence of Identity.** There exists $1 \in R$ such that $x \,.\, 1 = x = 1 \,.\, x \;\forall x \in R$.

(*iv*) **Existence of Inverse.** There exists $y \in R$, where $y = \frac{1}{x}$ such that $x \,.\, y = 1 = y \,.\, x \;\forall\; x \in R$.

(*c*) **Distributive Laws :** $\forall\, x, y, z \in R$, the following hold :

Left. $x \,.\, (y + z) = x \,.\, y + x \,.\, z$

Right. $(y + z) \,.\, x = y \,.\, x + z \,.\, x$

Hence R forms a field.

Example 3. *Prove that the set of all complex numbers forms a field.* (G.N.D.U. 1992 S)

Sol. Let C be the set of all complex numbers.

(*a*) **Under Addition :**

(*i*) **Commutative.** $x + y = y + x \;\forall\; x, y \in C$.

(*ii*) **Associative.** $(x + y) + z = x + (y + z) \;\forall\; x, y, z \in C$.

(*iii*) **Existence of Identity.** There exists $0 \in C$ such that $x + 0 = x = 0 + x \;\forall\; x \in C$.

(*iv*) **Existence of Inverse.** There exists $x' \in C$ such that $x + x' = 0 = x' + x \;\forall\; x \in C$.

(*b*) **Under Multiplication :**

(*i*) **Commutative.** $x \,.\, y = y \,.\, x \;\forall\; x, y \in C$

(*ii*) **Associative.** $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z) \;\forall\; x, y, z \in C$.

(*iii*) **Existence of Identity.** There exists $1 \in C$ such that $x \,.\, 1 = x = 1 \,.\, x \;\forall\; x \in C$.

(*iv*) **Existence of Inverse.** There exists $x' \in C$ such that $x \,.\, x' = 1 = x' . x \;\forall\; x \in C$.

(*c*) **Distributive Laws.** $\forall\, x, y, z \in C$, the following hold :

Left. $x \,.\, (y + z) = x \,.\, y + x \,.\, z$

Right. $(y + z) \,.\, x = y \,.\, x + z \,.\, x$

Hence C forms a field.

2

VECTOR SPACES

1. Binary Compositions.

There are two compositions *viz.*

(*i*) *Internal Composition*

(*ii*) *External Composition.*

(*i*) **Internal Composition. Def. 1.** *Let A be a set. Then the mapping $f : A \times A \to A$ is said to be internal composition in it.*

This is also called **binary composition.**

The mapping associates to each ordered pair $(a, b) \in A$, a unique member $f((a, b))$ of A, where $a, b \in A$.

For example : *Consider* **R**, *the set of all real numbers.*

Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be defined as :

$f((a, b)) = ab \; \forall \, (a, b) \in \mathbf{R} \times \mathbf{R} \, ; a, b \in \mathbf{R}$.

Then f is a composition in **R**.

Def. 2. *Let A be a set. If there exists a rule, denoted by $\oplus$, which associates to each ordered pair (a, b), $a, b \in A$, a unique element $a \oplus b$ of A, then $\oplus$ is said to be binary composition in A.*

In scalars, we use addition by '+' and multiplication by '.'

(*ii*) **External Composition. Def.** *Let V and F be any two non-empty sets. Then the mapping $f : V \times F \to V$ is said to be an external composition in V over F.*

(G.N.D.U. 1993)

2. Vector Space.

Def. *Let (F, +.) be the given field in which elements of F are scalars. Further let V, be a non-empty set, where the elements of V are vectors. Then V is said to be a vector space over the field F if it satisfies the following postulates :* (G.N.D.U 1993)

I. Under addition. The addition of vectors (denoted by '+') is defined as *internal composition* in V satisfying the following :

$\mathbf{V}_1$. **Closure.** $\forall \, \alpha, \beta \in V, \alpha + \beta \in V$.

$\mathbf{V}_2$. **Associativity.** $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \;\; \forall \; \alpha, \beta, \gamma \in V$.

$\mathbf{V}_3$. **Existence of Identity.** *There exists $0 \in V$ such that*

$$\alpha + \mathbf{0} = \alpha = \mathbf{0} + \alpha \quad \forall \; \alpha \in V.$$

Remember : 0 is called as **zero-vector** in V.

$\mathbf{V}_4$. **Existence of Inverse.** *There exists $-\alpha \in V \, \forall \, \alpha \in V$ such that*

$$(-\alpha) + \alpha = \mathbf{0} = \alpha + (-\alpha).$$

Remember : $-\alpha$ is the **negative** of α.

$\mathbf{V}_5$. **Commutativity.** $\alpha + \beta = \beta + \alpha \;\; \forall \, \alpha, \beta \in V$.

II. Second Scalar Multiplication. The scalar multiplication is defined as external composition in V over F satisfying the following :

$\mathbf{V}_6$. $\forall \, a \in F$ *and* $\forall \, \alpha \in V$, $a\alpha \in V$.

$\mathbf{V}_7$. $a(\alpha + \beta) = a\alpha + a\beta \quad \forall \, a \in F \quad$ *and* $\; \forall \, \alpha, \beta \in V$.

V_8. $(a+b)\alpha = a\alpha + b\alpha \quad \forall\ a, b \in F$ and $\forall\ \alpha \in V$.

V_9. $(ab)\alpha = a(b\alpha) \quad \forall\ a, b \in F$ and $\forall\ \alpha \in V$.

V_{10}. $1 . \alpha = \alpha \quad \forall \quad \alpha \in F$.

Remember : (*i*) 1 is the **unity element** in the field F.

(*ii*) **0** is the zero element of V and 0 is the zero element of F.

Notation. The vector space of V over the field F is denoted as V (F).

(*i*) V (F) is a *real vector space* if F is the field R of real numbers.

(*ii*) V (F) is a *rational vector space* if F is the field Q of rational numbers.

(*iii*) V (F) is a *complex vector space* if F is the field C of complex numbers.

PROPERTIES OF VECTOR SPACE

Let V be a vector space over a given field F.

Property I. $\alpha . \mathbf{0} = \mathbf{0}, \mathbf{0} \in V, \forall \alpha \in F$.

Proof. Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, where **0** is the zero element of V,

$\therefore \quad \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} \quad \forall \alpha \in F$

$\Rightarrow \quad \alpha\mathbf{0} + \alpha\mathbf{0} = \alpha\mathbf{0}$

$\Rightarrow \quad \alpha\mathbf{0} + \alpha\mathbf{0} = \alpha\mathbf{0} + \mathbf{0}$ [Note it]

Hence $\quad \alpha\mathbf{0} = \mathbf{0}$. [*By Cancellation Law*]

Property II. $0 . x = \mathbf{0} \quad \forall x \in V, 0 \in F, \mathbf{0} \in V$.

Proof. Since $0 + 0 = 0$, where 0 is the zero element of F.

$\therefore \quad (0+0)x = 0x \quad \forall x \in V$

$\Rightarrow \quad 0x + 0x = 0x$

$\Rightarrow \quad 0x + 0x = 0x + \mathbf{0}$ [Note it]

Hence $\quad 0x = \mathbf{0}$. [*By Cancellation Law*]

Property III. $(-\alpha)x = -(\alpha x) \quad \forall \alpha \in F$ and $x \in V$.

Proof. Since $\alpha \in F, \quad \therefore \ -\alpha \in F$

$\Rightarrow \quad \alpha + (-\alpha) = 0 \in F$.

Now $\quad \alpha x + (-\alpha)x = [\alpha + (-\alpha)]x \quad \forall x \in V$

$\Rightarrow \quad \alpha x + (-\alpha)x = 0x \quad \forall x \in V$

$\Rightarrow \quad \alpha x + (-\alpha)x = \mathbf{0} \quad \forall x \in V$ [*Property II*]

$\Rightarrow$ $(-\alpha)x$ is the additive inverse of αx in V.

Hence $\quad (-\alpha)x = -(\alpha x)$.

Property IV. $\alpha(-x) = -(\alpha x), \quad \forall \alpha \in F$ and $x \in V$.

Proof. Since $x \in V, \quad \therefore \ -x \in V$

$\Rightarrow \quad x + (-x) = \mathbf{0} \in V$.

Now $\quad \alpha x + \alpha(-x) = \alpha[x + (-x)] \quad \forall x \in V$

$\Rightarrow \quad \alpha x + \alpha(-x) = \alpha\mathbf{0} \quad \forall x \in V$

$\Rightarrow \quad \alpha x + \alpha(-x) = \mathbf{0} \quad \forall x \in V$ [*Property I*]

$\Rightarrow$ $\alpha(-x)$ is the additive inverse of αx in V.

Hence $\quad \alpha(-x) = -(\alpha x)$.

Property V. $\alpha(x-y) = \alpha x - \alpha y \quad \forall \alpha \in F$ and $x, y \in V$.

Proof. $\alpha(x-y) = \alpha[x + (-y)]$

$= \alpha x + \alpha(-y) = \alpha x - (\alpha y)$ [*Property IV*]

Hence $\quad \alpha(x-y) = \alpha x - \alpha y$.

Property VI. $\alpha x = 0,\ x \neq 0$

$\Rightarrow$ $\alpha = 0,$ *where* $x \in V, \alpha \in F.$

Proof. Let $\alpha x = 0,\ x \neq 0.$

If possible, let $\alpha \neq 0$. Then α^{-1} exists, where $\alpha^{-1} \in$ F.

$\therefore$ $\alpha^{-1}(\alpha x) = \alpha^{-1}(0)$

$\Rightarrow$ $(\alpha^{-1}\alpha)x = 0$

$\Rightarrow$ $1x = 0$ $[\because \alpha^{-1}\alpha = 1,$ *where* $\alpha \in F]$

$\Rightarrow$ $x = 0$, which is a contradiction.

Hence $\alpha = 0.$

Property VII. $\left.\begin{array}{l}(i)\ x+y=x+z \Rightarrow y=z \\ (ii)\ y+x=z+x \Rightarrow y=z\end{array}\right\} \forall x, y, z \in V.$

Proof. (*i*) $x + y = x + z$

$\Rightarrow$ $(-x) + (x + y) = (-x) + (x + z)$

[*Adding* $(-x)$ *on the left to both sides*]

$\Rightarrow$ $((-x) + x) + y = ((-x) + x) + z$ [*By Cancellation Law*]

$\Rightarrow$ $0 + y = 0 + z$

$\Rightarrow$ $y = z.$

(*ii*) $y + x = z + x$

$\Rightarrow$ $(y + x) + (-x) = (z + x) + (-x)$ [*Adding* $(-x)$ *on the right to both sides*]

$\Rightarrow$ $y + (x + (-x)) = z + (x + (-x))$

$\Rightarrow$ $y + 0 = z + 0$

$\Rightarrow$ $y = z.$

Hence the result.

SOLVED EXAMPLES

Example 1. *What is the zero vector in the vector space* $\mathbf{R}^4$?

Sol. (0, 0, 0, 0).

Example 2. *Which of the following sets form vector spaces over reals ? If not why ?*

(*i*) *The set of all rational over* **R**. (G.N.D.U. 1987)

(*ii*) $V = \{a + ib\ ;$ *for all* $a, b \in \mathbf{Z}\}$. (G.N.D.U. 1987)

(*iii*) *All polynomials over* **R** *with constant term zero.* (G.N.D.U. 1987)

(*iv*) *All polymials over* **R** *with constant term 1.* (G.N.D.U. 1998)

(*v*) *All polynomials with positive real coefficients.*

(*vi*) *All polynomials* $f(x)$ *over* **R** *such that* $f(1) = 0$.

(*vii*) *All polynomials* $f(x)$ *over* **R** *such that* $f(1) = 5$.

(*viii*) *All upper (lower) triangular matrices of order n over* **R**.

(*ix*) *All n-rowed symmetric* (*skew symmetric*) *matrices over* **C**. (G.N.D.U. 1987)

Sol. (*i*) No ; not closed under scalar multiplication.

(*ii*) No ; not closed under scalar multiplication.

(*iii*) Yes.

(*iv*) No ; not closed under addition.

(*v*) No ; zero element does not exist.

(*vi*) Yes.

(*vii*) No, not closed under addition.

(*viii*) Yes.

(*ix*) Yes.

Example 3. *Examine the truth or otherwise of the following statements :*

(*i*) *A vector space must have at least two elements.* (G.N.D.U 1985 S, 85)

(*ii*) *In the definition of a vector space $V(F)$, the axiom $1 . v = v$, for all $v \in V$ can be dropped.* (P.U. 1996 ; G.N.D.U. 1993, 86)

(*iii*) *A vector space has always an infinite number of elements.* (G.N.D.U. 1985 S)

Sol. (*i*) False ; A vector space may have only one element.

(*ii*) False (*iii*) False.

Example 4. *Does the set V of all ordered pairs of integers form a vector space over the field* **R** *of real numbers with addition and scalar multiplication defined as follows :*

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \text{ for } (a_1, a_2), (b_1, b_2) \in V$$
$$\alpha (a_1, a_2) = (\alpha a_1, \alpha a_2) \text{ for } \alpha \in \mathbf{R}, (a_1, a_2) \in V?$$

Sol. We have : $\mathbf{V} = \{(a_i, a_j) \ \forall a_i, a_j \in \mathbf{I}\}$.

Since V is not closed for scalar multiplication as defined *i.e.*, the product of an element of V by a scalar $\in \mathbf{R}$ may $\in$ V

$\Big[\because$ **For Ex.** $(5, 6) \in$ V and $\frac{1}{7} \in \mathbf{R}$,

$\therefore$ by scalar multiplication, as defined, we have

$$\frac{1}{7}(5, 6) = \left(\frac{5}{7}, \frac{6}{7}\right) \notin \mathbf{V} \text{ because } \frac{5}{7}, \frac{6}{7} \notin \mathbf{Z}\Big]$$

$\therefore$ **V(R)** is not a vector space.

Example 5. *Let V be the set of all pairs (a, b) of real numbers. Examine each of the following cases, where V is a vector space over* ***R*** *or not :*

(*i*) $(a, b) + (a', b') = (0, b + b')$; $\alpha (a, b) = (\alpha a, \alpha b)$

(*ii*) $(a, b) + (a', b') = (a + a', b + b')$; $\alpha (a, b) = (0, \alpha b)$

(*iii*) $(a, b) + (a', b') = (a + a', b + b')$; $\alpha (a, b) = (\alpha^2 a, \alpha^2 b)$

(*iv*) $(a, b) + (a', b') = (a, b)$; $\alpha (a, b) = (\alpha a, \alpha b)$,

where $a, b, a', b', \alpha \in \mathbf{R}$.

Sol. In order to show that V is not a vector space, we have only to show that one of the postulates of a vector space does not hold.

(*i*) Here there exists no additive identity *i.e.*, there exists no ordered pair $(c, d) \in$ V such that

$(c, d) + (a, b) = (a, b) \ \forall (a, b) \in$ V.

$$\left[\because (c, d) + (a, b) = (0, d + b) \neq (a, b)\right] \quad (by\ def.)$$

Thus $\mathbf{V}_3$ is not satisfied.

Hence **V (R)** is not a vector space.

(*ii*) Let $(a, b) \in$ V for $a \neq 0$, then

$$1 (a, b) = (0, 1b) = (0, b) \neq (a, b) \quad (Def.)$$

Thus $\mathbf{V}_{10}$ is not satisfied.

Hence **V (R)** is not a vector space.

(*iii*) By def., $(\alpha+\beta)(a, b) = ((\alpha+\beta)^2 a, (\alpha+\beta)^2 b)$ for $\alpha, \beta \in \mathbf{R}$

and $\alpha(a, b)+\beta(a, b) = (\alpha^2 a, \alpha^2 b)+(\beta^2 a, \beta^2 b)$

$= (\alpha^2 a+\beta^2 a, \alpha^2 b+\beta^2 b)$ (*Def.*)

$= ((\alpha^2+\beta^2)a, (\alpha^2+\beta^2)b)$ [*By Distributive Law in* **R**]

Since $(\alpha+\beta)^2 \neq \alpha^2+\beta^2 \ \forall \alpha, \beta \in \mathbf{R}$,

$\therefore$ $(\alpha+\beta)(a, b) \neq \alpha(a, b)+\beta(a, b)$.

Thus $\mathbf{V}_8$ is not satisfied.

Hence **V (R)** is not a vector space.

(*iv*) By def., $(a, b)+(a', b') = (a, b)$

and $(a', b')+(a, b) = (a', b')$

$\Rightarrow$ $(a, b)+(a', b') \neq (a', b')+(a, b)$

Thus $\mathbf{V}_5$ is not satisfied.

Hence **V(R)** is not a vector space.

Example 6. *If* $(F, +, .)$ *is a field, then show that* **F** (F) *is a vector space and deduce that*

(*i*) **C** *is a vector space over field* **C**

(*ii*) **R** *is a vector space over field* **R**

(*iii*) **C** *is a vector space over field* **R**

(*iv*) **R** *is not a vector space over field* **C**.

Sol. In order to prove that **F (F)** is a vector space, we have to verify all the postulates for the vector space.

Since (F, +, .) is a field, (*Given*)

$\therefore$ (F, +) is an abelian group.

$\therefore$ $\mathbf{V}_1 - \mathbf{V}_5$ are satisfied.

The scalar multiplication is the same as multiplication of field because the set F and field are same, so

$\mathbf{V}_6$. $\alpha(x+y) = \alpha x+\alpha y \quad \forall \alpha \in F$ and $\forall x, y \in F$

[$\because$ *Elements of the field are distributive*]

$\mathbf{V}_7$. $(\alpha+\beta)x = \alpha x+\beta x \quad \forall \alpha, \beta \in F$ and $\forall x \in F$

$\mathbf{V}_8$. $\alpha(\beta x) = (\alpha\beta)x \quad \forall \alpha, \beta \in F$ and $\forall x \in F$

[$\because$ *Elements of the field are associative for multiplication*]

$\mathbf{V}_9$. $1(x) = x$ for $1 \in F$, $\forall x \in F$

[$\because$ *1 is the multiplicative identity of F*]

Hence **F** (F) is a vector space.

(*i*) Since (**C**, +, .) is a field,

$\therefore$ as proved above **C (C)** is a vector space.

(*ii*) Since (**R**, +, .) is a field,

$\therefore$ as proved above **R (R)** is a vector space.

(*iii*) In order to prove that **C (R)** is a vector space, we have to verify all the postulates for the vector space.

Since (**C**, +, .) is a field,

$\therefore$ (**C**, +) is an abelian group.

$\therefore$ $\mathbf{V}_1 - \mathbf{V}_5$ are satisfied.

$\mathbf{V}_6$. Since the product of a real number by a complex number is again a complex number, so

$\alpha \in \mathbf{R}, x \in \mathbf{C} \Rightarrow \alpha x \in \mathbf{C}$.

Thus closure property is satisfied.

Since the elements of **R** are also elements of **C** (*i.e.*, $\mathbf{R} \subset \mathbf{C}$) and elements of **C** are distributive w.r.t. addition and multiplication, so

$\mathbf{V_7}$. $\alpha(x+y) = \alpha x + \alpha y$ $\quad \forall \alpha \in \mathbf{R}$ and $\forall x, y \in \mathbf{C}$.

$\mathbf{V_8}$. $(\alpha+\beta)x = \alpha x + \beta x$ $\quad \forall \alpha, \beta \in \mathbf{R}$ and $\forall x \in \mathbf{C}$.

$\mathbf{V_9}$. $\alpha(\beta x) = (\alpha\beta)x$ $\quad \forall \alpha, \beta \in \mathbf{R}$ and $\forall x \in \mathbf{C}$.

i.e., the elements of **C** are associative for multiplication.

$\mathbf{V_{10}}$. $1.(x) = x$ $\quad$ for $1 \in \mathbf{R}$ and $\forall x \in \mathbf{C}$.

i.e., $1 \in \mathbf{R}$ is the multiplicative identity of **C**.

Hence **C (R)** is a vector space.

(*iv*) Since product of complex number by a real number is a complex number and not a real number *i.e.*,

$$\alpha \in \mathbf{C},\ x \in \mathbf{R} \Rightarrow \alpha x \in \mathbf{C} \text{ but } \alpha x \notin \mathbf{R},$$

$\therefore$ **R (C)** is not closed for scalar multiplication.

Hence **R (C)** is not a vector space.

Example 7. *If R is the field of real numbers and V is the set of vectors in a plane. Further if addition of vectors is the internal binary composition in V and the multiplication of the elements of R with those of V as the external composition, prove that V* **(R)** *is a vector space.*

Sol. Given :

$$\mathbf{V} = \{ (x, y) \mid x, y \in \mathrm{R} \}$$

[$\because$ *V is a set of vectors in a plane,* $\therefore$ *elements are ordered pairs*]

Let us define the *addition of vectors* in V as

$$(x, y) + (x', y') = (x + x', y + y')$$

and the *scalar multiplication of* $\alpha \in \mathrm{R}$ and $(x, y) \in \mathrm{V}$ as

$$\alpha(x, y) = (\alpha x, \alpha y).$$

I. Under Addition :

$\mathbf{V_1}$. **Closure.** $(x_1 + x_2, y_1 + y_2) \in \mathrm{V}$

$\forall (x_1, y_1), (x_2, y_2) \in \mathrm{V}$

[$\because x_1, y_1; x_2, y_2 \in R \Rightarrow x_1 + x_2,\ y_1 + y_2 \in R$ *and* $(x_1, y_1) + (x_2 y_2) = (x_1 + x_2, y_1 + y_2)$]

$\mathbf{V_2}$. **Associativity.**

$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$

$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathrm{V}$.

Proof. $((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$ [*def.*]

$= (x_1 + x_2 + x_3,\ y_1 + y_2 + y_3)$ [*def.*]

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$ [*Associativity in R*]

$= (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$.

$\mathbf{V_3}$. **Existence of Identity.** $\forall (x, y) \in \mathrm{V}, \exists (0, 0) \in \mathrm{V}$ *such that*

$(x, y) + (0, 0) = (x, y) = (0, 0) + (x, y)$.

Proof. $(x, y) + (0, 0) = (x + 0, y + 0)$ [*def.*]

$= (x, y)$

Similarly $(0, 0) + (x, y) = (x, y)$.

Thus $(x, y) + (0, 0) = (x, y) = (0, 0) + (x, y)$.

Here $(0, 0)$ is the **zero element** of V.

V_4. Existence of Inverse. $\forall (x, y) \in V, \exists (-x, -y) \in V$ *such that*

$$(-x, -y) + (x, y) = (0, 0) = (x, y) + (-x, -y).$$

Proof. $(-x, -y) + (x, y) = ((-x) + x, (-y) + y)$ [*def.*]

$= (0, 0)$.

Similarly $(x, y) + (-x, -y) = (0, 0)$.

$\therefore \quad (-x, -y) + (x, y) = (0, 0) = (x, y) + (-x, -y)$.

Here $(-x, -y)$ is the **inverse element** of (x, y) in **V**.

V_5. Commutativity. $(x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1)$

$\forall (x_1, y_1), (x_2, y_2) \in V$.

Proof. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ [*def.*]

$= (x_2 + x_1, y_2 + y_1)$

[$\because$ $x_1, x_2; y_1, y_2 \in R$ *and commutative law holds in R*]

[*def.*]

$= (x_2, y_2) + (x_1, y_1)$.

II. Under Scalar Multiplication :

V_6. $\forall \alpha \in R$ and $\forall (x, y) \in V, x, y \in R$

$\alpha (x, y) = (\alpha x, \alpha y) \in V$. [$\because \alpha x, \alpha y \in R$]

V_7. $\alpha ((x_1, y_1) + (x_2, y_2)) = \alpha (x_1, y_1) + \alpha (x_2, y_2)$

$\forall \alpha \in R$ and $\forall (x_1, y_1), (x_2, y_2) \in V$.

Proof. $\alpha ((x_1, y_1) + (x_2, y_2)) = \alpha (x_1 + x_2, y_1 + y_2)$ [*def.*]

$= (\alpha (x_1 + x_2), \alpha (y_1 + y_2))$ [$\because$ *of* V_6]

$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2) = (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$ [*def.*]

$= \alpha (x_1, y_1) + \alpha (x_2, y_2)$. [$\because$ *of* V_6]

V_8. $(\alpha + \beta)(x, y) = \alpha (x, y) + \beta (x, y) \ \forall \alpha, \beta \in R$ and $\forall (x, y) \in V$.

Proof. $(\alpha + \beta)(x, y) = ((\alpha + \beta) x, (\alpha + \beta) y)$ [$\because$ *of* V_5]

$= (\alpha x + \beta x, \alpha y + \beta y)$ [$\because$ *of* V_7]

$= (\alpha x, \alpha y) + (\beta x, \beta y)$ [*def.*]

$= \alpha (x, y) + \beta (x, y)$, [$\because$ *of* V_6]

V_9. $(\alpha . \beta)(x, y) = \alpha (\beta (x, y)) \ \forall \alpha, \beta \in R$ and $\forall (x, y) \in V$.

Proof. $(\alpha . \beta)(x, y) = ((\alpha . \beta) x, (\alpha . \beta) y)$ [$\because$ *of* V_6]

$= (\alpha (\beta x), \alpha (\beta y)) = \alpha (\beta x, \beta y)$ [$\because$ *of* V_6]

$= \alpha (\beta (x, y))$.

V_{10}. $1 (x, y) = (x, y) \ \forall (x, y) \in V$ and $1 \in R$. [$\because$ *of* V_6]

Proof. $1 (x, y) = (1. x\ 1. y)$ [*def.*]

$= (x, y)$.

Hence V (R) is a vector space.

Example 8. *Prove that the set $V_n(F)$ of all ordered n-tuples of all elements of any field F is a vector space with vector addition and scalar multiplication defined as below :*

$$\forall\, x = (x_1, x_2, \ldots\ldots, x_n) \text{ and } y = (y_1, y_2, \ldots\ldots, y_n) \in V_n(F)$$

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n),$$

where $x_1, x_2, \ldots\ldots, x_n;\ y_1, y_2, \ldots\ldots, y_n \in F$ *and* $\forall\, \alpha \in F$, *we define*

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots\ldots \alpha x_n).$$

Sol. Here

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$$

$$\forall\, x = (x_1, x_2, \ldots\ldots, x_n), y = (y_1, y_2, \ldots\ldots, y_n) \text{ in } V_n(F),$$

where $x_1, x_2, \ldots\ldots, x_n; y_1, y_2, \ldots\ldots, y_n \in F$

and $\alpha x = (\alpha x_1, \alpha x_2, \ldots\ldots, \alpha x_n)\ \forall\, \alpha \in F.$

I. Under Addition :

V_1. Closure. $(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n) \in V_n(F)$

$\forall\, (x_1, x_2, \ldots\ldots, x_n), (y_1, y_2, \ldots\ldots y_n) \in V_n(F)$

$[\because x_1, y_1; x_2, y_2; \ldots\ldots; x_n, y_n \in F \Rightarrow x_1 + y_1; x_2 + y_2; \ldots\ldots; x_n + y_n \in F$

and $(x_1, x_2, \ldots\ldots, x_n) + (y_1, y_2, \ldots\ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)]$

$\therefore\ x + y \in V_n(F)\ \forall\, x, y \in V_n(F).$

V_2. Associativity.

$$((x_1, x_2, \ldots\ldots, x_n) + (y_1, y_2, \ldots\ldots, y_n)) + (z_1, z_2, \ldots\ldots, z_n)$$
$$= (x_1, x_2, \ldots\ldots, x_n) + ((y_1, y_2, \ldots\ldots, y_n) + (z_1, z_2, \ldots\ldots, z_n))$$

$\forall\ (x_1, x_2, \ldots\ldots, x_n), (y_1, y_2, \ldots\ldots, y_n), (z_1, z_2, \ldots\ldots, z_n) \in V_n(F).$

Proof. $((x_1, x_2, \ldots\ldots, x_n) + (y_1, y_2, \ldots\ldots, y_n)) + (z_1, z_2, \ldots\ldots, z_n)$

$= (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n) + (z_1, z_2, \ldots\ldots, z_n)$ [*def.*]

$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots\ldots, x_n + y_n + z_n)$ [*def.*]

$= (x_1, x_2, \ldots, x_n) + ((y_1 + z_1), (y_2 + z_2) + \ldots\ldots, (y_n + z_n))$ [*Associativity in F*]

$= (x_1, x_2, \ldots\ldots, x_n) + ((y_1, y_2, \ldots\ldots, y_n) + (z_1, z_2, \ldots\ldots, z_n))$

$\therefore\ (x + y) + z = x + (y + z)\ \forall\, x, y, z \in V_n(F).$

V_3. Existence of Identity. $\forall\, (x_1, x_2, \ldots\ldots, x_n) \in V_n(F), \exists\ (0, 0, \ldots\ldots, 0)\ \forall\, V_n(F)$ *such that*

$$(x_1, x_2, \ldots\ldots, x_n) + (0, 0, \ldots\ldots, 0) = (x_1, x_2, \ldots\ldots, x_n)$$
$$= (0, 0, \ldots\ldots, 0) + (x_1, x_2, \ldots\ldots, x_n).$$

Proof. $(x_1, x_2, \ldots\ldots, x_n) + (0, 0, \ldots\ldots, 0) = (x_1 + 0, x_2 + 0, \ldots\ldots, x_n + 0)$ [*def.*]

$= (x_1, x_2, \ldots\ldots, x_n).$

Similarly $(0, 0, \ldots\ldots, 0) + (x_1, x_2, \ldots\ldots, x_n) = (x_1, x_2, \ldots\ldots, x_n)$

$\therefore\ (x_1, x_2, \ldots\ldots, x_n) + (0, 0, \ldots\ldots, 0) = (x_1, x_2, \ldots\ldots, x_n)$

$= (0, 0, \ldots\ldots, 0) + (x_1, x_2, \ldots\ldots, x_n)$

Here $(0, 0, \ldots\ldots, 0)$ is the **zero element** of $V_n(F)$.

Thus $x + \mathbf{0} = x = \mathbf{0} + x\ \ \forall\, x \in V_n(F)$ and $\mathbf{0} \in V_n(F)$.

V_4. Existence of Inverse.

$\forall\, x = (x_1, x_2, \ldots\ldots, x_n) \in V_n(F), \exists$

$-x = (-x_1, -x_2, \ldots\ldots, -x_n) \in V_n(F)$ such that

$(-x) + x = 0 = x + (-x)$.

Proof. $(-x) + x = (-x_1, -x_2, \ldots\ldots, -x_n) + (x_1, x_2, \ldots\ldots, x_n)$

$= (-x_1 + x_1, -x_2 + x_2, \ldots\ldots, -x_n + x_n)$ [*def.*]

$= (0, 0, \ldots\ldots, 0) = 0$

Similarly $x + (-x) = 0$.

$\therefore\ (-x) + x = 0 = x + (-x)$.

Here $-x$ is the inverse element of x in $V_n(F)$.

V_5. Commutativity. $x + y = y + x\ \ \forall\, x, y \in V_n(F)$.

Proof. $x + y = (x_1, x_2, \ldots\ldots, x_n) + (y_1, y_2, \ldots\ldots, y_n)$

$= (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$ [*def.*]

$= (y_1 + x_1, y_2 + x_2, \ldots\ldots, y_n + x_n)$

[$\because$ $x_1, x_2, \ldots\ldots, x_n$; $y_1, y_2, \ldots\ldots, y_n \in F$ *and commutative law holds in F*]

$= (y_1, y_2, \ldots\ldots, y_n) + (x_1, x_2, \ldots\ldots, x_n)$ [*def.*]

$= y + x$.

II. Under Scalar Multiplication.

V_6. $\forall\, \alpha \in F$ *and* $\forall\, x = (x_1, x_2, \ldots\ldots, x_n) \in V_n(F)$,

where $x_1, x_2, \ldots\ldots, x_n \in F, \alpha x \in V_n(F)$.

Proof. $x_1, x_2, \ldots\ldots, x_n \in F$, and $\alpha \in F$.

$\Rightarrow$ $\alpha x_1, \alpha x_2, \ldots\ldots, \alpha_n x_n \in F$ [$\because$ *F is a field*]

$\Rightarrow$ $(\alpha x_1, \alpha x_2, \ldots\ldots, \alpha x_n) \in V_n(F)$

$\Rightarrow$ $\alpha x \in V_n(F)$.

V_7. $\alpha(x + y) = \alpha x + \alpha y\ \ \forall\, \alpha \in F$ *and* $\forall\, x, y \in V_n(F)$.

Proof. $\alpha(x + y) = \alpha((x_1, x_2, \ldots\ldots, x_n) + (y_1, y_2, \ldots\ldots, y_n))$

$= \alpha(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$ [*def.*]

$= (\alpha(x_1 + y_1), \alpha(x_2 + y_2), \ldots\ldots, \alpha(x_n + y_n))$ [$\because$ *of* V_6]

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots\ldots, \alpha x_n + \alpha y_n)$

$= (\alpha x_1, \alpha x_2, \ldots\ldots, \alpha x_n) + (\alpha y_1, \alpha y_2, \ldots\ldots, \alpha y_n)$ [*def.*]

$= \alpha(x_1, x_2, \ldots\ldots, x_n) + \alpha(y_1, y_2, \ldots\ldots, y_n)$ [$\because$ *of* V_6]

$= \alpha x + \alpha y$.

V_8. $(\alpha + \beta)x = \alpha x + \beta x\ \ \forall\, \alpha, \beta \in F,\ \forall\, x \in V_n(F)$.

Proof. $(\alpha + \beta)x = (\alpha + \beta)(x_1, x_2, \ldots\ldots, x_n)$

$= ((\alpha + \beta)x_1, (\alpha + \beta)x_2, \ldots\ldots, (\alpha + \beta)x_n)$ [$\because$ *of* V_6]

$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \ldots\ldots, \alpha x_n + \beta x_n)$ [$\because$ *of* V_7]

$= (\alpha x_1, \alpha x_2, \ldots\ldots, \alpha x_n) + (\beta x_1, \beta x_2, \ldots\ldots, \beta x_n)$ [*def.*]

$$= \alpha (x_1 , x_2 ,......, x_n) + \beta (x_1 , x_2 ,......, x_n) \qquad [\because \text{ of } V]$$
$$= \alpha x + \beta x.$$

V₉. $(\alpha\beta)\, x = \alpha(\beta x) \;\forall\, \alpha, \beta \in \mathrm{F},\; \forall\, x \in \mathrm{V}_n (\mathrm{F})$.

Proof. $(\alpha\beta)\, x = (\alpha\beta)\, (x_1 , x_2 ,......, x_n)$

$$= ((\alpha\beta)x_1 , (\alpha\beta)x_2 ,......, (\alpha\beta)x_n) \qquad [\because \text{ of } V_6]$$
$$= (\alpha(\beta x_1) , \alpha(\beta x_2),......, \alpha(\beta x_n))$$
$$= \alpha\, (\beta x_1 , \beta x_2 ,......, \beta x_n)$$
$$= \alpha\, (\beta\, (x_1 , x_2 ,......, x_n)) \qquad [\because \text{ of } V_6]$$
$$= \alpha(\beta x). \qquad [\because \text{ of } V_6]$$

V₁₀. $1 . x = x \;\forall\, x \in \mathrm{V}_n (\mathrm{F})$, *and* $1 \in \mathrm{F}$.

Proof. $1 . x = 1 . (x_1 , x_2 ,......, x_n)$

$$= (1x_1 , 1x_2 ,......, 1x_n) \qquad [def.]$$
$$= (x_1 , x_2 ,......, x_n)$$
$$= x.$$

Hence V_n (F) is a vector space.

Example 9. *If V is the set of all real-valued continuous (differentiable or integrable) functions defined in some interval [0, 1]. Then show that V* **(R)** *is a vector space with addition and scalar multiplication defined as follows* :

$$(f + g)x = f(x) + g(x) \;\; \forall f, g \in V$$

and $$(\alpha f)x = \alpha f(x) \qquad \forall\, \alpha \in \mathbf{R},\, f \in V.$$

Sol. We have $\mathrm{V} = \{f \mid f(x) \in \mathbf{R} \text{ for } x \in [0, 1]\}$.

I. Under Addition :

V₁. Closure. Since the sum of two continuous functions is also a continuous function and the sum of two real values is also a real value,

$\therefore$ $$(f + g)x = f(x) + g(x)$$

asserts that $(f + g)$ is also a real-valued continuous function.

$\therefore$ $$f + g \in \mathrm{V} \;\forall\, f, g \in \mathrm{V}.$$

V₂. Associativity. $(f + g) + h = f + (g + h) \;\forall f, g, h \in V$.

Proof. $[(f + g) + h]\, (x) = (f + g)\, (x) + h\, (x) \qquad [def.]$

$$= \{f(x) + g(x)\} + h(x) \qquad [def.]$$
$$= f(x) + \{g(x) + h(x)\} \qquad [\textit{By associativity for addition in } R]$$
$$= f(x) + (g + h)\, (x) \qquad [def.]$$
$$= [f + (g + h)]\, (x) \;\forall\, x \in [0, 1].$$

V₃. Existence of Identity. First of all, let us define **0** as below :

$$\mathbf{0}\, (x) = 0 \in \mathrm{R} \;\forall\, x \in [0, 1],$$

then **0** is real-valued continuous function $\in$ V.

So $\exists\, \mathbf{0} \in$ V such that

$$(\mathbf{0} + f)x = \mathbf{0}(x) + f(x) \qquad [def.]$$
$$= 0 + f(x) = f(x) \;\; \forall\, x \in [0, 1] \qquad [\because 0 \textit{ is additive identity in } R]$$

Thus $\mathbf{0} + f = f$.

Similarly $f + \mathbf{0} = f$.

Here $\mathbf{0}$ is the additive identity.

V_4. **Existence of Inverse.** For each $f \in V$, let us define $-f$ as

$$(-f)(x) = -f(x),$$

then $-f$ is also a real-valued continuous function $\in V$.

$$\forall f \in V,\ \exists -f \in V, \text{ such that}$$

$$\begin{aligned}[f + (-f)](x) &= f(x) + (-f)(x) && [def.]\\ &= f(x) + \{-f(x)\}\\ &= 0, \text{ additive identity in R}\\ &= \mathbf{0}(x)\ \forall x \in [0, 1].\end{aligned}$$

$\therefore$ $f + (-f) = \mathbf{0}$.

Similarly $(-f) + f = \mathbf{0}$.

Here $-f$ is the additive inverse of f.

V_5. **Commutativity.** $f + g = g + f \quad \forall f, g \in V$.

Proof.
$$\begin{aligned}(f + g)(x) &= f(x) + g(x) && [def.]\\ &= g(x) + f(x) && [\textit{Commutativity under addition in } \mathbf{R}]\\ &= (g + f)(x)\quad \forall x \in [0, 1] && [def.]\end{aligned}$$

II. Under Scalar Multiplication :

V_6. $\forall \alpha \in \mathbf{R},\ \forall f \in V,$

$$(\alpha f)(x) = \alpha f(x) \in \mathbf{R}$$

$\Rightarrow$ $\alpha f \in V$.

V_7. $\alpha(f + g) = \alpha f + \alpha g \quad \forall \alpha \in \mathbf{R},\ \forall f, g \in V$.

Proof.
$$\begin{aligned}[\alpha(f + g)](x) &= \alpha[(f + g)](x)\ \forall x \in [0, 1] && [def.]\\ &= \alpha[f(x) + g(x)] && [def.]\\ &= \alpha f(x) + \alpha g(x)\\ &= (\alpha f)(x) + (\alpha g)(x) && [\because \text{ of } V_6]\\ &= (\alpha f + \alpha g)(x)\ \forall x \in [0, 1]. && [def.]\end{aligned}$$

V_8. $(\alpha + \beta) f = \alpha f + \beta f \quad \forall \alpha, \beta \in \mathbf{R},\ \forall f \in V$.

Proof.
$$\begin{aligned}[(\alpha + \beta) f](x) &= (\alpha + \beta) f(x)\ \forall \alpha \in [0, 1]\\ &= \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x)\\ &= (\alpha f + \beta f)(x)\ \forall x \in [0, 1]\end{aligned}$$

V_9. $(\alpha\beta) f = \alpha(\beta f) \quad \forall \alpha, \beta \in \mathbf{R},\ \forall f \in V$.

Proof.
$$\begin{aligned}[(\alpha\beta) f](x) &= (\alpha\beta) f(x) = \alpha[\beta f(x)]\\ &= \alpha[(\beta f)(x)]\\ &= [\alpha(\beta f)(x)]\ \forall x \in [0, 1].\end{aligned}$$

V_{10}. $1f = f\ \forall f \in V, \quad 1 \in \mathbf{R}$.

Proof.
$$\begin{aligned}(1f)(x) &= 1f(x)\\ &= f(x)\ \forall x \in [0, 1] && [\because 1 \in \mathbf{R} \textit{ is multiplicative identity}]\end{aligned}$$

Hence V (**R**) is a vector space.

Example 10. *Let V be set of all real-valued continuous functions defined as [0, 1] such that* $f\left(\frac{2}{3}\right) = 2$. *Show that V is not a vector space over R (reals) under addition and scalar multiplication defined as :*

$$(f+g)(x) = f(x) + g(x) \quad \forall f, g \in V$$
$$(\alpha f)(x) = \alpha f(x) \quad \forall \alpha \in R, f \in V.$$

Sol. We have

$$V = \left\{ f \mid f \text{ is real valued continuous function defined on } [0, 1] \text{ such that } f\left(\frac{2}{3}\right) = 2 \right\}.$$

Let $f, g \in$ V. By def., f, g are real valued continuous functions defined on [0, 1] such that $f\left(\frac{2}{3}\right) = 2$ and $g\left(\frac{2}{3}\right) = 2$.

Now $(f+g)\left(\frac{2}{3}\right) = f\left(\frac{2}{3}\right) + g\left(\frac{2}{3}\right)$

$= 2 + 2 = 4$

$\Rightarrow \quad f + g \notin$ V

$\Rightarrow$ Closure property does not hold.

Hence V is not a vector space over R.

Example 11. *If* P(x) *denotes the set of all ploynomials in one indeterminate x over a field F, then show that* P (x) *is a vector space over F with addition defined as addition of polynomials and scalar multiplication defined as the product of polynomial by an element of F.*

Sol. We have :

$$\mathbf{P}(x) = \{p(x) \mid p(x) = a_0 + a_1 x + \ldots\ldots + a_n x^n + \ldots\ldots\} = \left\{ \sum_{n=0}^{\infty} a_n x^n \text{ for } a\text{'s} \in F \right\}$$

Let us define the *addition* as :

If $\quad p(x) = \Sigma\, a_n x^n \in \mathbf{P}(x)$,

$\quad q(x) = \Sigma\, b_n x^n \in \mathbf{P}(x)$,

then $\quad p(x) + q(x) = \Sigma\,(a_n + b_n)\, x^n. \quad$ ...(1)

Let us define *scalar multiplication* as :

If $\quad p(x) = \Sigma\, a_n x^n \in \mathbf{P}(x),\ \alpha \in \mathbf{F}$,

then $\quad \alpha\, p(x) = \Sigma\,(\alpha a_n)\, x^n \quad$ ...(2)

I. Under Addition :

V_1. Closure. $\forall\, p(x), q(x) \in \mathbf{P}(x)$

$\Rightarrow \quad p(x) + q(x) = \Sigma\,(a_n + b_n)\, x^n \in \mathbf{P}(x)$

[$\because$ *if* $a_n, b_n \in F \Rightarrow a_n + b_n \in F$ *as field is closed for addition*]

V_2. Associativity. $\forall\, p(x) = \Sigma\, a_n x^n,\ q(x) = \Sigma\, b_n x^n,\ r(x) = \Sigma\, c_n x^n \in \mathbf{P}(x)$,

we have $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$.

Proof. $(p(x)+q(x))+r(x)=\{\Sigma a_n x^n+\Sigma b_n x^n\}+\Sigma c_n x^n$

$=\{\Sigma(a_n+b_n)x^n\}+\Sigma c_n x^n$ [*By* (*1*)]

$=\Sigma((a_n+b_n)+c_n)x^n$ [*By* (*1*)]

$=\Sigma(a_n+(b_n+c_n))x^n$

[$\because$ *if* $a_n, b_n, c_n \in F \Rightarrow (a_n+b_n)+c_n=a_n+(b_n+c_n)$]

$=\Sigma a_n x^n+(\Sigma b_n x^n+\Sigma c_n x^n)$

$=p(x)+(q(x)+r(x))$.

V_3. Existence of Identity. $\forall\, p(x) \in P(x)$, $\exists\, 0(x) \in P(x)$ *such that*

$$0(x)+p(x)=p(x)+0(x).$$

Proof. Here $0(x)=\Sigma 0\, x^n \in F(x)$

Now $0(x)+p(x)=\Sigma 0\, x^n+\Sigma a_n x^n \;\forall\, p(x) \in P(x)$

$=\Sigma(0+a_n)x^n$

$=\Sigma a_n x^n$ [$\because$ *0* $\in F$ *is the additive identity in F*]

Similarly $p(x)+0(x)=p(x)$

$\therefore$ $0(x)+p(x)=p(x)=p(x)+0(x)$.

Thus $0(x)$ is the additive identity of $P(x)$.

V_4. Existence of Inverse. $\forall\, p(x)=\Sigma a_n x^n \in P(x)$,

$$\exists -p(x)=\Sigma(-a_n)x^n \in P(x)$$

such that $p(x)+(-p(x))=0(x)=(-p(x))+p(x)$.

Proof. $p(x)+(-p(x))=\Sigma a_n x^n+\Sigma(-a_n)x^n$

$=\Sigma(a_n+(-a_n))x^n$ [*By* (*1*)]

$=\Sigma 0\, x^n$ [$\because -a_n$ *is the additive inverse of* a_n *in F*]

$=0(x)$.

Similarly $(-p(x))+p(x)=0(x)$

$\therefore$ $p(x)+(-p(x))=0(x)=(-p(x))+p(x)$.

Thus additive inverse of each element in $P(x)$ exists.

V_5. Commutativity. $p(x)+q(x)=q(x)+p(x) \;\forall\, p(x), q(x) \in P(x)$.

Proof. $p(x)+q(x)=\Sigma a_n x^n+\Sigma b_n x^n$

$=\Sigma(a_n+b_n)x^n$ [*By* (*1*)]

$=\Sigma(b_n+a_n)x^n$ [$\because$ *Elements of F are commutative for addition*]

$=\Sigma b_n x^n+\Sigma a_n x^n=q(x)+p(x)$.

Thus elements of $P(x)$ are commutative for addition.

II. Under Scalar Multiplication :

V_6. $\forall\, \alpha \in F$, *and* $\forall\, p(x) \in P(x)$,

$$\alpha\, p(x)=\alpha\, \Sigma a_n x^n=\Sigma(\alpha a_n)x^n \in P(x)$$

[$\because$ $\alpha \in F$, $a_n \in F \Rightarrow \alpha a_n \in F$ *because F is closed for multiplication*]

V₇. $\alpha(p(x)+q(x)) = \alpha p(x) + \alpha q(x)\ \forall\ \alpha \in \mathrm{F}$ *and* $\forall\ p(x), q(x) \in \mathbf{P}(x)$.

Proof. $\alpha(p(x)+q(x)) = \alpha\,(\Sigma\, a_n x^n + \Sigma\, b_n x^n)$

$= \alpha(\Sigma(a_n + b_n)\,x^n)$ [*By* (*1*)]

$= \Sigma\alpha(a_n + b_n)\,x^n$ [*By* (*2*)]

$= \Sigma(\alpha a_n + \alpha b_n)\,x^n$ [$\because$ *Elements of F satisfy distributive law*]

$= \Sigma\,(\alpha a_n)x^n + \Sigma\,(\alpha b_n)\,x^n$ [*By* (*1*)]

$= \alpha\,\Sigma\, a_n x^n + \alpha\Sigma b_n x^n$ [*By* (*2*)]

$= \alpha p\,(x) + \alpha q(x)$.

V₈. $(\alpha+\beta)\,p(x) = \alpha p(x) + \beta p(x)\ \ \forall\, \alpha, \beta \in \mathrm{F}$,

and $\forall\, p(x) \in \mathbf{P}(x)$.

Proof. $(\alpha+\beta)\,p(x) = (\alpha+\beta)\,\Sigma\, a_n\, x_n$

$= \Sigma((\alpha+\beta)\,a_n)x^n$ [*By* (*2*)]

$= \Sigma(\alpha a_n + \beta a_n)\,x^n$ [$\because$ *Elements of F satisfy distributive law*]

$= \Sigma\alpha a_n\, x^n + \Sigma\,\beta a_n\, x^n$ [*By* (*1*)]

$= \alpha\Sigma a_n\, x^n + \beta\Sigma a_n\, x^n$ [*By* (*2*)]

$= \alpha p(x) + \beta p(x)$.

V₉. $(\alpha\,.\,\beta)\,p(x) = \alpha\,(\beta p(x))\ \forall\, \alpha, \beta \in \mathrm{F}$ *and* $\forall\ p(x) \in \mathbf{P}(x)$.

Proof. $(\alpha\,.\,\beta)\,p(x) = (\alpha\beta)\Sigma a_n\, x^n$

$= \Sigma((\alpha\beta)a_n)x^n$ [*By* (*2*)]

$= \Sigma(\alpha\,(\beta a_n)x^n\,)$ [*By* (*2*)]

$= \alpha\,(\beta\,(\Sigma a_n x^n\,))$ [*By* (*2*)]

$= \alpha\,(\beta\,\Sigma p(x))$.

V₁₀. $1.\,p(x) = p(x)\ \forall\, p(x)\ \in \mathbf{P}(x)$ *and* $1 \in \mathrm{F}$.

Proof. $1.\,p(x) = 1.\ \Sigma a_n\, x^n = \ \Sigma\,(1 a_n)x^n$ [*By* (*2*)]

$=\ \Sigma a_n\, x^n$ [$\because$ *1 is the multiplicative identity of F*]

$= p(x)$.

Hence $\mathbf{P}(x)$ is a vector space.

Example 12. *Let V_n (F) denote the set of all polynomials in x of degree $\leq n$ (a non-negative integer) and the zero polynomial. Prove that V_n (F) is a vector space over the field F under the usual addition and scalar multiplication of polynomials.* (Pbi. U. 1986)

Sol. We have

$\mathrm{V}_n\,(\mathrm{F}) = \{\, f(x)\ |\ f(x)$ is a polynomial of degree $\leq n$ or $f(x)$ is a zero polynomial$\}$.

Let us define the *addition* and *scalar multiplication* as :

If $f(x) = a_0 + a_1\, x + \ldots\ldots + a_k\, x^k$

and $g\,(x) = b_0 + b_1\, x\ + \ldots\ldots + b_k\, x^k$,

where either $f(x)$ is a zero polynomial or deg $f(x) \leq n$

and $g\,(x)$ is a zero polynomial or deg $g\,(x) \leq n$.

Then $f(x)+g(x) = (a_0+b_0)+(a_1+b_1)x+\ldots\ldots$

and $\alpha f(x) = (\alpha a_0)+(\alpha a_1)x+\ldots\ldots+(\alpha a_k)x^k$, where $\alpha \in F$.

I. Under Addition :

V_1. Closure. Let $f(x),\ g(x) \in V_n(F)$.

Now $f(x)+g(x) = (a_0+b_0)+(a_1+b_1)x+\ldots\ldots$

$\in V_n(F)$ [$\because$ *$f(x)+g(x)$ is either a zero polynomial or a polynomial of degree $\le n$*]

$\Rightarrow$ $V_n(F)$ is closed under addition.

$V_2 - V_5$ are similar to those in Ex. 11.

Under Scalar Multiplication :

V_6. $\forall\, \alpha \in F$ and $f(x) \in V_n(F)$

$\alpha f(x) = (\alpha a_0)+(\alpha a_1)x+\ldots\ldots+(\alpha a_k)x^k$

$\in V_n$ [$\because$ *$\alpha f(x)$ is either a zero polynomial or a polynomial of degree $\le n$*]

$V_7 - V_{10}$ are similar to those in Ex. 11.

Here $V_n(F)$ is a vector space.

Example 13. *Let F be a field and V the set of all $m \times n$ matrices over the field F. The addition of matrices is defined as internal composition and multiplication of any scalar with a matrix as the external composition. Prove that $V(F)$ is a vector space.* (G.N.D.U. 1986, 85)

Sol. Let $A = [a_{ij}]_{m\times n},\ B = [b_{ij}]_{m\times n},\ C = [c_{ij}]_{m\times n} \in V(F)$,

where $a_{ij}, b_{ij}, c_{ij} \in F$.

I. Under Addition :

V_1. Closure. $\forall\, A, B \in V(F), A+B \in V(F)$.

Proof. $A+B = [a_{ij}]_{m\times n}+[b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n} \in V(F)$.

V_2. Associativity. $(A+B)+C = A+(B+C)\ \forall\, A, B, C \in V(F)$.

Proof. $(A+B)+C = ([a_{ij}]_{m\times n}+[b_{ij}]_{m\times n})+[c_{ij}]_{m\times n}$

$= [(a_{ij}+b_{ij})_{m\times n}]+[c_{ij}]_{m\times n} = [(a_{ij}+b_{ij})+c_{ij}]_{m\times n}$

$= [a_{ij}+(b_{ij}+c_{ij})]_{m\times n}$ [$\because$ *Associativity under addition in F*]

$= [a_{ij}]_{m\times n}+[b_{ij}+c_{ij}]_{m\times n} = [a_{ij}]_{m\times n}+([b_{ij}]_{m\times n}+[c_{ij}]_{m\times n})$

$= A+(B+C)$.

V_3. Existence of Identity. $A+O = A = A+O\ \forall\, A \in V(F), O \in V(F)$.

Proof. $A+O = [a_{ij}]_{m\times n}+[0]_{m\times n} = [a_{ij}+0]_{m\times n} = [a_{ij}]_{m\times n} = A$.

Similarly $O+A = A$.

Thus $A+O = A = O+A$.

Here $O = O_{m\times n} = [0]_{m\times n}$ is the **identity.**

V_4. Existence of Inverse. $\forall\, A \in V(F)$,

there exists $-A \in V(F)$ such that $A+(-A) = O = (-A)+A$.

Proof. $A+(-A) = [a_{ij}]_{m\times n}+[-a_{ij}]_{m\times n} = [a_{ij}+(-a_{ij})]_{m\times n} = [0]_{m\times n} = O$.

Similarly $(-A)+A = O$.

Thus $A + (-A) = O = (-A) + A$.

Here $-A$ is the inverse of A.

V_5. Commutativity. $A + B = B + A \quad \forall A, B \in V(F)$.

Proof. $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$

$= [b_{ij} + a_{ij}]_{m \times n}$ [*Commutativity under addition in F*]

$= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A.$

II. Under Scalar Multiplication :

V_6. $\forall \alpha \in F, \forall A \in V(F)$.

$$\alpha A = \alpha [a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n} \in V(F).$$

V_7. $(\alpha + \beta) A = \alpha A + \beta A \quad \forall \alpha, \beta \in F, \forall A \in V(F)$.

Proof. $(\alpha + \beta) A = (\alpha + \beta) [a_{ij}]_{m \times n} = [(\alpha + \beta) a_{ij}]_{m \times n} = [\alpha a_{ij} + \beta a_{ij}]_{m \times n}$

$= [\alpha a_{ij}]_{m \times n} + [\beta a_{ij}]_{m \times n} = \alpha [a_{ij}]_{m \times n} + \beta [a_{ij}]_{m \times n}$

$= \alpha A + \beta A.$

V_8. $\alpha (A + B) = \alpha A + \alpha B \quad \forall \alpha \in F, \forall A, B \in V(F)$.

Proof. $\alpha (A + B) = \alpha ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) = \alpha [a_{ij} + b_{ij}]_{m \times n} = [\alpha (a_{ij} + b_{ij})]_{m \times n}$

$= [\alpha a_{ij} + \alpha b_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n} + [\alpha b_{ij}]_{m \times n} = \alpha [a_{ij}]_{m \times n} + \alpha [b_{ij}]_{m \times n}$

$= \alpha A + \alpha B.$

V_9. $(\alpha\beta) A = \alpha(\beta A) \quad \forall \alpha, \beta \in F, \forall A \in V(F)$.

Proof. $(\alpha\beta) A = (\alpha\beta) [a_{ij}]_{m \times n} = [(\alpha\beta) a_{ij}]_{m \times n} = [\alpha (\beta a_{ij})]_{m \times n} = \alpha [\beta a_{ij}]_{m \times n}$

$= \alpha (\beta [a_{ij}]_{m \times n}) = \alpha (\beta A).$

V_{10}. $1. A = A \quad \forall A \in V(F), \; 1 \in F.$

Proof. $1. A = 1 [a_{ij}]_{m \times n} = [1. a_{ij}]_{m \times n} = [a_{ij}]_{m \times n} = A.$

Hence V(F) is a vector space.

Example 14. *Show that the set M of $n \times n$ diagonal matrices over the field of reals is a vector space under the usual addition of matrices and the scalar multiplication of a matrix by a real number.*

Sol. We have : $M = \{A \mid A = [a_{ij}]_{n \times n}, a_{ij} = 0 \text{ for } i \neq j; a_{ii} \in R\}$.

Let $A, B, C \in M$, where $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$, $C = [c_{ij}]_{n \times n}$.

Then $a_{ij} = b_{ij} = c_{ij} = 0$ for $i \neq j; 1 \leq i, j \leq n$ and $a_{ii}, b_{ii}, c_{ii} \in R$.

Under Addition :

V_1. Closure. Let $A, B \in M$.

$\therefore \; a_{ij} = 0$ for $i \neq j$ and $b_{ij} = 0$ for $i \neq j$

and $a_{ii}, b_{ii} \in R$

$\therefore \; A + B = [a_{ij}]_{n \times n} + [b_{ij}]_{n \times n}$

$= [a_{ij} + b_{ij}]_{n \times n}$ [$\because \; a_{ij} + b_{ij} = 0 + 0 = 0$ *for* $i \neq j$ *and* $a_{ii} + b_{ii} \in R$]

$\in M$

$\therefore$ M is closed under addition.

$V_2 - V_5$ are same as in Ex. 13.

Under Scalar Multiplication :

V_6. Let $\alpha \in F$ and $A \in M$.

Now $A \in M \Rightarrow a_{ij} = 0$ for $i \neq j$ and $a_{ii} \in R$

$\therefore$ $\alpha A = \alpha [a_{ij}]_{n \times n}$

$= [\alpha a_{ij}]_{n \times n}$, $\quad$ [$\because \alpha a_{ij} = \alpha (0) = 0$ *for* $i \neq j$ *and* $\alpha a_{ii} \in R$]

$\in M$

$\therefore$ M is closed under scalar multiplication.

$V_7 - V_{10}$ are same as in Ex. 13.

Hence M is a vector space over F.

Example 15. *Show that $V = \{ i\beta \mid \beta \in R\}$ is a vector space over the field R where addition of the vector space and the scalar multiplication of the elements of V by those of R are respectively the addition of complex numbers and the multiplication of a real number with a complex number.*

Sol. We have : $V = \{ i\beta \mid \beta \in R \}$, where $i = \sqrt{-1}$.

Let $x, y \in V$. Then $x = ia$, $y = ib$, where $a, b \in R$.

We define *addition* and *scalar multiplication* as :

$x + y = i(a + b)$ and $tx = i(ta)$ for $t \in R$.

I. Under Addition :

V_1. Closure.

Let $x, y \in V$.

Then $x + y = ia + ib = i(a + b) \in V$

$\therefore$ V is closed under addition.

V_2. Associativity.

Let $x, y, z \in V$ *i.e.* $x = ia$, $y = ib$, $z = ic$; $a, b, c \in R$.

Then $x + (y + z) = ia + (ib + ic) = ia + i(b + c) = i[a + (b + c)]$

$= i[(a + b) + c]$ $\quad$ [$\because$ *Associative Law holds in R*]

$= i(a + b) + ic = (ia + ib) + ic = (x + y) + z.$

$\therefore$ Associative Law holds in V.

V_3. Existence of Identity.

Let $x \in V$ *i.e.*, $x = ia$; $a \in R$.

Then $O = i0 \in V$

Now $O + x = i0 + ia = i(0 + a) = ia = x$

Similarly $x + O = x.$

$\therefore$ $O + x = x = x + O$

$\therefore$ O is the additive identity.

V_4. Existence of Inverse.

Let $x \in V$ *i.e.*, $x = ia$; $a \in R$.

Then $-x = -(ia) = i(-a) \in V$ $\quad$ [$\because a \in R \Rightarrow -a \in R$]

Now $x + (-x) = ia + i(-a)$

$= i[a + (-a)] = i(0) = O$

Similarly $(-x) + x = O.$

$\therefore$ $-x = i(-a)$ is the additive inverse of $x = ia$ in V.

V_5. Commutativity. Let $x, y \in V$ *i.e.*, $x = ia$, $y = ib$, $a, b \in R$.

Now $x + y = ia + ib$

$$= i(a + b) = i(b + a) \quad [\because \textit{Commutative Law holds in } R]$$

$$= ib + ia = y + x.$$

$\therefore$ addition is commutative in V.

II. Under Scalar Multiplication :

V_6. $\forall\, t \in R$ and $x \in V$.

Then $tx = t(ia) = i(ta) \in V$

$\therefore$ V is closed under scalar multiplication.

V_7. Let $t, s \in R$ and $x \in V$.

Then $(t + s)x = (t + s)(ia) = i[(t + s)a] = i[ta + sa] = i(ta) + i(sa) = t(ia) + s(ia)$

$$= tx + sx.$$

V_8. Let $t \in R$ and $x, y \in V$.

Then $t(x + y) = t(ia + ib) = t(i(a + b)) = i[t(a + b)] = i[ta + tb] = i(ta) + i(tb) = t(ia) + t(ib)$

$$= tx + ty.$$

V_9. Let $t, s \in R$ and $x \in V$.

Then $(ts)x = (ts)(ia) = i[(ts)a] = i[t(sa)] = t[i(sa)] = t[s(ia)] = t(sx)$.

V_{10}. Let 1 be unity element of R and $x \in V$.

Then $1.x = 1.(ia) = i(1.a) = i(a) = x$.

Hence V is a vector space over R.

Example 16. *Show that the set of all elements of type*

$$a + b\sqrt{2} + c\sqrt[3]{3}\ ;\ a, b, c \in Q$$

form a vector space over the field Q under usual addition and scalar multiplication of real numbers.

Sol. We have $V = \{a + b\sqrt{2} + c\sqrt[3]{3} \mid a, b, c \in Q\}$.

Let $x = a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}$, $y = a_2 + b_2\sqrt{2} + c_3\sqrt[3]{3}$; $a_1, b_1, c_1 ; a_2, b_2, c_2 \in Q$.

We define *addition* and *scalar multiplication* as :

$$x + y = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}$$

and $$\alpha x = (\alpha a_1) + (\alpha b_1)\sqrt{2} + (\alpha c_1)\sqrt[3]{3}.$$

I. Under Addition :

V_1. Closure. Let $x, y \in V$.

Then $x + y = (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})$

$$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}$$

$$\in V \quad [\because a_1, b_1, c_1 ; a_2, b_2, c_2 \in Q \Rightarrow a_1 + a_2, b_1 + b_2, c_1 + c_2 \in Q]$$

V_2. Associativity.

Let $x, y, z \in V$.

Then $x + (y + z) = (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + [(a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3}) + (a_3 + b_3\sqrt{2} + c_3\sqrt[3]{3})]$

$$= (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3})\,[(a_2 + a_3) + (b_2 + b_3)\sqrt{2} + (c_2 + c_3)\sqrt[3]{3}]$$

$$= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3))\sqrt{2} + (c_1 + (c_2 + c_3))\sqrt[3]{3}$$
$$= ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3)\sqrt{2} + ((c_1 + c_2) + c_3)\sqrt[3]{3}$$

[$\because$ *Associative Law holds in Q*]

$\therefore$ Associative law holds in V.

V_3. Existence of Identity.

Let $x \in V$ *i.e.*, $x = a + b\sqrt{2} + c\sqrt[3]{3}$; $a, b, c \in Q$.

Then $\quad O = 0 + 0\sqrt{2} + 0\sqrt[3]{3} \in V$

Now $\quad O + x = (0 + 0\sqrt{2} + 0\sqrt[3]{3}) + (a + b\sqrt{2} + c\sqrt[3]{3}) = (0 + a) + (0 + b)\sqrt{2} + (0 + c)\sqrt[3]{3}$

$$= a + b\sqrt{2} + c\sqrt[3]{3} = x.$$

Similarly $x + O = x$.

$\therefore \quad O + x = x = x + O$.

$\therefore \quad O = 0 + 0\sqrt{2} + 0\sqrt[3]{3}$ is the additive identity in V.

V_4. Existence of Inverse.

Let $x \in V$ *i.e.*, $x = a + b\sqrt{2} + c\sqrt[3]{3}$; $a, b, c \in Q$.

Then $-x = (-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3} \in Q$

[$\because a, b, c \in Q \Rightarrow -a, -v, -c \in Q$]

Now $\quad x + (-x) = (a + b\sqrt{2} + c\sqrt[3]{3}) + ((-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3})$

$$= (a + (-a)) + (b + (-b))\sqrt{2} + (c + (-c))\sqrt[3]{3} = 0 + 0\sqrt{2} + 0\sqrt[3]{3} = O.$$

Similarly $(-x) + x = O$.

$\therefore \quad x + (-x) = O = (-x) + x$.

$\therefore \quad -x = (-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3}$ is additive inverse of $x = a + b\sqrt{2} + c\sqrt[3]{3}$ in V.

V_5. Commutativity. Let $x, y \in V$.

Then $\quad x + y = (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})$

$$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}$$
$$= (a_2 + a_1) + (b_2 + b_1)\sqrt{2} + (c_2 + c_1)\sqrt[3]{3} \qquad [\because \textit{Associative Law holds in } Q]$$
$$= (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3}) + (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) = y + x$$

$\therefore$ Addition is commutative in V.

II. Under Scalar Multiplication :

V_6. Let $\alpha \in Q$ and $x \in V$.

Then $\alpha x = \alpha(a + b\sqrt{2} + c\sqrt[3]{3}) = (\alpha a) + (\alpha b)\sqrt{2} + (\alpha c)\sqrt[3]{3}$

$\in V \qquad$ [$\because \alpha, a \in Q \Rightarrow \alpha a \in Q$; etc.]

$\therefore \quad$ V is closed under scalar multiplication.

V_7. Let $\alpha, \beta \in Q$ and $x \in V$.

Then $(\alpha+\beta)x = (\alpha+\beta)(a+b\sqrt{2}+c\sqrt[3]{3}) = (\alpha+\beta)a + (\alpha+\beta)b\sqrt{2} + (\alpha+\beta)c\sqrt[3]{3}$

$= (\alpha a+\beta a) + (\alpha b+\beta b)\sqrt{2} + (\alpha c+\beta c)\sqrt[3]{3}$

$= (\alpha a+\alpha b\sqrt{2}+\alpha c\sqrt[3]{3}) + (\beta a+\beta b\sqrt{2}+\beta c\sqrt[3]{3})$

$= \alpha(a+b\sqrt{2}+c\sqrt[3]{3}) + \beta(a+b\sqrt{2}+c\sqrt[3]{3})$

$= \alpha x+\beta x.$

V_8. Let $\alpha \in Q$ and $x, y \in V$.

Then $\alpha(x+y) = \alpha[(a_1+b_1\sqrt{2}+c_1\sqrt[3]{3}) + (a_2+b_2\sqrt{2}+c_2\sqrt[3]{3})]$

$= \alpha[(a_1+a_2)+(b_1+b_2)\sqrt{2}+(c_1+c_2)\sqrt[3]{3}]$

$= [\alpha(a_1+a_2)] + [\alpha(b_1+b_2)]\sqrt{2} + [\alpha(c_1+c_2)]\sqrt[3]{3}$

$= (\alpha a_1+\alpha a_2) + (\alpha b_1+\alpha b_2)\sqrt{2} + (\alpha c_1+\alpha c_2)\sqrt[3]{3}$

$= (\alpha a_1+\alpha b_1\sqrt{2}+\alpha c_1\sqrt[3]{3}) + (\alpha a_2+\alpha b_2\sqrt{2}+\alpha c_2\sqrt[3]{3})$

$= \alpha(a_1+b_1\sqrt{2}+c_1\sqrt[3]{3}) + \alpha(a_2+b_2\sqrt{2}+c_2\sqrt[3]{3})$

$= \alpha x+\alpha y.$

V_9. Let $\alpha, \beta \in Q$ and $x \in V$.

Then $(\alpha\beta)x = (\alpha\beta)[a+b\sqrt{2}+c\sqrt[3]{3}] = [(\alpha\beta)a] + [(\alpha\beta)b]\sqrt{2} + [(\alpha\beta)c]\sqrt[3]{3}$

$= [\alpha(\beta a)] + [\alpha(\beta b)]\sqrt{2} + [\alpha(\beta c)]\sqrt[3]{3}$

$= \alpha[\beta a+\beta b\sqrt{2}+\beta c\sqrt[3]{3}] = \alpha[\beta(a+b\sqrt{2}+c\sqrt[3]{3})]$

$= \alpha(\beta x).$

V_{10}. Let 1 be the unity element of Q and $x \in V$.

Then $1 . x = 1.(a+b\sqrt{2}+c\sqrt[3]{3}) = (1.a) + (1.b)\sqrt{2} + (1.c)\sqrt[3]{3} = a+b\sqrt{2}+c\sqrt[3]{3}$

$= x.$

Hence V is a vector space over Q.

3. Vector Sub-spaces.

Def. *Let V be a vector space over the field F, then the non-empty subset W of V is said to be the vector sub-space if W itself is a vector space over F under the operations of V.*

Thus (I) V is a sub-space of V [$\because$ *V is a subset of itself and is a vector space under operations of V*]

(II) $\{0\}$ is a sub-space of V. [$\because$ $\{0\}$ *is a sub-set of V and is a vector space under operations of V*]

These two vector sub-spaces of V *viz.* $\{0\}$ and V itself are called **trivial sub-spaces.**

All other vector sub-spaces of V are called **non-trivial sub-spaces.**

For Example. *Consider* $V = \mathbf{V_3(R)} = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \mathbf{R}\}$.

We can easily see that it is a vector space for addition and scalar multiplication of vectors.

Consider $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbf{R}\}$.

We can also easily see that it is a vector space for addition and scalar multiplication of vectors.

Further since $W \subset V$, therefore, W is a vector sub-space of V.

THEOREMS

Theorem I. *A non-empty sub-set W of a vector space V(F) is a sub-space of V iff W is closed under vector addition and scalar multiplication.* (G.N.D.U. 1985 S)

Proof. Since W is a sub-space of V (F), [*Given*]

$\therefore$ W is closed under vector addition and scalar multiplication.

Hence the result.

Conversely. Given. W is a closed under vector addition and scalar multiplication.

To prove. W is a vector sub-space of V.

(I) Since W is closed under scalar multiplication, [*Given*]

$$\therefore \quad \forall x \in W, -1 \in F \Rightarrow (-1)x \in W$$
$$\Rightarrow -x \in W$$

[$\because$ $\forall x \in W \Rightarrow x \in V$ *and* $(-1)(x) = -x \in V$]

Thus additive inverse of each element of W exists.

(II) Since W is closed under vector addition, [*Given*]

$$\therefore \quad \forall x \in W, \; -x \in W \Rightarrow x + (-x) \in W$$

[$\because$ $x \in W \Rightarrow -x \in W \Rightarrow x + (-x) = \mathbf{0} \in V$]

$$\Rightarrow \mathbf{0} \in W$$

Thus **0** is the additive identity of W.

(III) Since elements of W are elements of V,

$\therefore$ vector addition is commutative and associative in W.

Thus W is an abelian group under vector addition.

Further W is closed under scalar multiplication and, therefore, the remaining properties of vector space also hold in W because they hold in V.

Hence W is a vector space and is a vector sub-space of V.

Theorem II. *A sub-set W of a vector space V(F) is a sub-space of V iff*

(*i*) *W is non-empty*

(*ii*) $\forall x, y \in W \Rightarrow x - y \in W$

(*iii*) $\forall \alpha \in F, x \in W \Rightarrow \alpha x \in W$.

Proof. (*i*) Since W is a sub-space of V(F),

$\therefore$ (W, +) is an abelian group and is a sub-group of (V, +).

$\therefore$ W possesses the additive identity **0**.

Hence W is non-empty.

(*ii*) $\forall x, y \in W \Rightarrow x - y \in W$ [$\because$ $y \in W \Rightarrow -y \in W$]

$\Rightarrow x + (-y) \in W$ [*By Closure Property under addition*]

$\Rightarrow x - y \in W$.

(*iii*) $\forall \alpha \in F, x \in W$, where W is a vector space

$\therefore$ $\alpha x \in W$. [*By def.*]

Conversely. Given. (*i*), (*ii*) and (*iii*) hold.

To prove. W is a vector sub-space of V (F).

To prove this, all the properties for a vector space must hold.

$\forall x \in W \Rightarrow x \in V$ [$\because W \subset V$]

Also $1 \in F \Rightarrow -1 \in F$

$\therefore$ $\forall y \in W$ and $-1 \in F \Rightarrow (-1)y \in W$ [*By (iii)*]

$\Rightarrow -y \in W$

Thus inverse of every element of W exists.

$\forall x \in W$ and $y \in W$

$\Rightarrow x \in W$ and $-y \in W \Rightarrow x-(-y) \in W$ [$\because$ *By (ii)*]

$\Rightarrow x+y \in W$

Thus W is closed under addition.

$\forall x \in W$ and $-x \in W \Rightarrow x+(-x) \in W$

$\Rightarrow 0 \in W.$

Thus identity exists in W under addition.

The remaining properties also hold for W. [$\because W \subset V$]

Hence W is a vector sub-space of V(F).

Theorem III. *A sub-set W of a vector space V(F) is a sub-space of V(F) iff* $\forall \alpha, \beta \in F$ *and* $\forall x, y \in W \Rightarrow \alpha x+\beta y \in W.$ (P.U. 1985)

Proof. Given. W is a sub-space of V(F).

To prove. $\forall \alpha, \beta \in F$ *and* $\forall x, y \in W \Rightarrow \alpha x+\beta y \in W$

$$\left.\begin{array}{ll} & \alpha \in F, x \in W \Rightarrow \alpha x \in W \\ \text{and} & \beta \in F, y \in W \Rightarrow \beta y \in W \end{array}\right\}$$ [$\because$ *W is closed under scalar multiplication*]

Now $\alpha x \in W, \beta y \in W \Rightarrow \alpha x+\beta y \in W$ [$\because$ *W is closed under addition*]

Hence $\forall \alpha, \beta \in F$ and $\forall x, y \in W \Rightarrow \alpha x+\beta y \in W.$

Conversely. Given. $\forall \alpha, \beta \in F$ and $\forall x, y \in W$

$\Rightarrow \alpha x+\beta y \in W.$

To prove. W is a sub-space of V(F).

Let us take $\alpha = 1$ and $\beta = -1.$

Here $1.x+(-1)y \in W \Rightarrow x-y \in W.$

Let us take $\beta = 0.$

Here $\alpha x+0.y \in W \Rightarrow \alpha x+\mathbf{0} \in W$ [$\because 0y = \mathbf{0}$]

$\Rightarrow \alpha x \in W.$

Hence W is a sub-space of V(F).

Theorem IV. *The intersection of the sub-spaces* W_1 *and* W_2 *of a vector space V(F) is also a sub-space.* (G.N.D.U. 1998, 96)

Proof. Let $x, y \in W_1 \cap W_2$ and $\alpha, \beta \in F.$

Now $x \in W_1 \cap W_2 \Rightarrow x \in W_1$ and $x \in W_2$

and $y \in W_1 \cap W_2 \Rightarrow y \in W_1$ and $y \in W_2$.

Since $x, y \in W_1$ and W_2 is a sub-space of V,

$\therefore \quad \alpha x + \beta y \in W_1$ ...(1) *[Th. III]*

Similarly $\alpha x + \beta y \in W_2$. ...(2)

(1) and (2) $\Rightarrow \alpha x + \beta y \in W_1 \cap W_2$.

Hence $W_1 \cap W_2$ is a sub-space of V(F).

Remark. The union of the sub-spaces may not be a sub-space.

For Ex. *Consider $V_3(F)$ to be a sub-space and W_1, W_2 be its two sub-spaces with elements of the type $(a, 0, 0)$ and $(0, b, 0)$ respectively.*

Let $x = (a, 0, 0) \in W_1$ and $y = (0, b, 0) \in W_2$.

If α, β are two non-zero scalars, then

$$\alpha x + \beta y = \alpha(a, 0, 0) + \beta(0, b, 0) = (\alpha a, \beta b, 0)$$

$\Rightarrow \quad \alpha x + \beta y \notin W_1$ and $\alpha x + \beta y \notin W_2$

$\Rightarrow \quad \alpha x + \beta y \notin W_1 \cap W_2$.

Hence $W_1 \cap W_2$ is not a vector-space.

SOLVED EXAMPLES

Example 1. *If W_1 and W_2 are sub-spaces of $V(F)$, prove that*

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2 \}$$

is a sub-space of V. (P.U. 1985 S)

Sol. If $x_1, x_2 \in W_1$; $y_1, y_2 \in W_2$

so that $x = x_1 + y_1, y = x_2 + y_2 \in W_1 + W_2$.

Now $\alpha, \beta \in F$ and $x_1, x_2 \in W_1$

$\therefore \quad \alpha x_1 + \beta x_2 \in W_1$ [$\because$ *W_1 is a sub-space*]

and $\alpha, \beta \in F$ and $y_1, y_2 \in W_2$

$\Rightarrow \quad \alpha y_1 + \beta y_2 \in W_2$ [$\because$ *W_2 is a sub-space*]

Thus $(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) \in W_1 + W_2$ ...(1)

Also for $\alpha, \beta \in F, x, y \in W_1 + W_2$

$\Rightarrow \quad \alpha x + \beta y = \alpha(x_1 + y_1) + \beta(x_2 + y_2) = \alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2 = (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2)$

$\in W_1 + W_2$. [*Using (1)*]

Hence $W_1 + W_2$ is a sub-space of V.

Example 2. *Prove that the intersection of an arbitrary family $\{W_\alpha : \alpha \in V\}$ of sub-spaces of a vector-space, V is a sub-space of V. Does the similar result hold for union ? Justify.* (G.N.D.U. 1989)

Sol. (*i*) Let $W_1, W_2, \ldots\ldots, W_n$ be n sub-spaces of a vector space V.

Let $W = \cap W_\alpha ; \alpha = 1, 2, \ldots, n$

Since $\mathbf{0} \in W_\alpha \quad \therefore \mathbf{0} \in \cap W_\alpha$

$\Rightarrow \mathbf{0} \in W$

$\Rightarrow$ W is non-empty.

Let $a, b \in F$ and $x, y \in W$, then $x, y \in$ each W_α

$\Rightarrow \quad ax + by \in$ each W_α, which is a sub-space

$\Rightarrow \quad ax + by \in \cap W_\alpha$

$\Rightarrow \quad ax + by \in W.$

Hence $W = \cap W_\alpha$ is a sub-space.

(*ii*) No. See Ex. 3.

Example 3. *Prove that union of two sub-spaces of a vector space is sub-space if and only if they are comparable.* (G.N.D.U. 1993, 85 ; P.U. 1989)

Or

Prove that the union of two sub-spaces W_1 and W_2 is a sub-space iff one of them is a subset of the other. (P.U. 1997)

Sol. Let $W_1 \subset W_2$. Then $W_1 \cup W_2 = W_2$.

Since W_2 is a sub-space, [*Given*]

$\therefore \quad W_1 \cup W_2$ is also a sub-space.

Again let $W_2 \subset W_1$. Then $W_1 \cup W_2 = W_1$.

Since W_1 is a sub-space, [*Given*]

$\therefore \quad W_1 \cup W_2$ is also a sub-space.

Conversely. Given. Let $W_1 \cup W_2$ be a sub-space.

To prove. Either $W_1 \subset W_2$ or $W_2 \subset W_1$ *i.e.*, W_1 and W_2 are comparable.

Proof. Let us assume that W_1 is not a sub-set of W_2 and W_2 is not a sub-set of W_1.

Since $\quad W_1 \not\subset W_2, \quad \therefore \quad \exists\, \alpha \in W_1, \alpha \notin W_2$ ...(1)

and $\quad W_2 \not\subset W_1, \quad \therefore \quad \exists\, \beta \in W_2, \beta \notin W_1$ ...(2)

From (1), $\alpha \in W_1 \cup W_2$ [$\because \alpha \in W_1$]

From (2), $\beta \in W_1 \cup W_2$ [$\because \beta \in W_2$]

But $W_1 \cup W_2$ is a sub-space [*Given*]

$\therefore \quad \alpha + \beta$ also $\in W_1 \cup W_2$

$\Rightarrow \quad \alpha + \beta \in W_1$ or W_2.

Let $\alpha + \beta \in W_1$. Also $\alpha \in W_1$ and W_1 is a sub-space

$\therefore \quad (\alpha + \beta) - \alpha \in W_1 \quad \Rightarrow \quad \beta \in W_1.$

But from (2), $\quad \beta \notin W_1.$

Thus there is a contradiction.

Hence either $W_1 \subset W_2$ or $W_2 \subset W_1$ *i.e.*, W_1 and W_2 are comparable.

Example 4. *Discuss whether or not R^2 is a sub-space of R^3.*

Sol. We know that $R^2 = \{(a, b) \mid a, b \in R\}$ is a vector space over R under usual addition and scalar multiplication of ordered pairs.

Also $\quad R^3 = \{(a, b, c) \mid a, b, c \in R\}$ is a vector space over R under usual addition and scalar multiplication of 3-triples.

But $\quad R^2$ is not a sub-set of R^3 [$\because (a, b) \in R^2 \Rightarrow (a, b) \notin R^3$]

Hence $\quad R^2$ is not a sub-space of R^3.

Example 5. *Let V be a vector space of polynomials of degree ≤ 6. Which of the following are sub-spaces ? Justify your answers. In each case $f(x)$ belongs to V.*

(*i*) $W = \{f(x) \mid f(0) = 1\}$

(*ii*) $W = \{f(x) \mid \deg f(x) \leq 4\}$

(*iii*) $W = \{ f(x) \mid f(1) = 0, f(3) = 0\}$

(*iv*) $W = \{ f(x) \mid$ *coeff. of* x^2 *is* 1 *or* $-1\}$

(*v*) $W = \{ f(x) \mid$ *having + ve coefficients*$\}$. (G.N.D.U. 1988 S)

Sol. (*i*) W is not a sub-space of V, because zero polynomial does not belong to W.

(*ii*) W is a sub-space of V.

Clearly W is a non-empty sub-set of V.

Consider $f(x), g(x) \in W$, then $\deg f(x) \leq 4$, $\deg g(x) \leq 4$.

For all $\alpha, \beta \in F$,

$\alpha f(x) + \beta g(x)$ is a polynomial of degree ≤ 4

$\Rightarrow \alpha f(x) + \beta g(x) \in W \quad \Rightarrow$ W is a sub-space of V.

(*iii*) W is a sub-space of V.

Let $f(x) = x^2 - 4x + 3$, which is a polynomial in x of degree 2 and $f(1) = f(3) = 0$, so $f(x) \in W$ and thus W is non-empty sub-set of V.

As in part (*b*), we can prove that W is a sub-space of V.

(*iv*) W is not a sub-space of V, because zero polynomial does not belong to W.

(*v*) W is not a sub-space of V, because zero polynomial does not belong to W.

Example 6. *Let V(R) be a vector space of all functions from R to R. Show that W_1 and W_2 are sub-spaces of V(R),*

where W_1 = set of all even function i.e. $\{f \mid f \in V$ and $f(-x) = f(x)\}$

and W_2 = set of all odd functions i.e. $\{f \mid f \in V$ and $f(-x) = -f(x)\}$.

Also check that $V = W_1 + W_2$.

Sol. (*i*) We have $W_1 = \{f \mid f \in V \text{ and } f(-x) = f(x)\}$.

Clearly $f \in W_1 \Rightarrow W_1$ is not empty $\quad [\because f(x) = x^2 + x^4 \in W_1]$

Now let $\alpha, \beta \in R$ and $f, g \in W_1$

so that $f(-x) = f(x)$ and $g(-x) = g(x)\ \forall x \in R$.

$$\therefore \quad (\alpha f + \beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x) = \alpha . f(-x) + \beta . g(-x) = \alpha f(x) + \beta g(x)$$
$$= (\alpha f)(x) + (\beta g)(x) = (\alpha f + \beta g)(x)$$

$\Rightarrow \quad \alpha f + \beta g \in W_1$.

Hence W_1 is a sub-space of V over R.

(*ii*) **Exactly similar to part (*i*).**

(*iii*) **To prove :** $V = W_1 + W_2$.

Let $f \in V$.

$$\text{Then} \quad f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \quad \forall x \in R$$

$$= F(x) + G(x), \quad \text{where } \frac{1}{2}(f(x) + f(-x)) = F(x)$$

$$\text{and } \frac{1}{2}(f(x) - f(-x)) = G(x)$$

Clearly $F(-x) = F(x)$ and $G(-x) = -G(x)$

$\Rightarrow$ $F(x) \in W_1$ and $G(x) \in W_2$

Thus $f = F + G$, where $F \in W_1$ and $G \in W_2$.

Hence $V = W_1 + W_2$.

Example 7. *Let V be a vector space in R^3. Examine whether the following are sub-spaces or not :*

(*i*) $W = \{(a, b, c) \mid a \geq 0\}$

(*ii*) $W = \{(a, b, c) \mid c$ *is an integer*$\}$

(*iii*) $W = \{(a, b, c) \mid a, b, c \in \mathbf{Q}\}$

(*iv*) $W = \{(a, b, c) \mid a \leq b \leq c\}$

(*v*) $W = \{(a, b, c) \mid a^2 + b^2 + c^2 \leq 1\}$

(*vi*) $W = \{(a, b, c) \mid a - 3b + 4c = 0\}$

(*vii*) $W = \{(a, b, c) \mid b + 4c = 0\}$

(*viii*) $W = \{(a, b, c) \mid a - b + c = 0, 2a + 3b - c = 0\}$. **(P.U. 1995)**

Sol. (*i*) Let $(a, b, c) \in W$ and $-5 \in \mathbf{R}$, then

$-5(a, b, c) = (-5a, -5b, -5c) \notin W$ $[\because -5a < 0]$

Thus V is not closed for scalar multiplication.

Hence W is not a sub-space of V.

(*ii*) Let $(a, b, c) \in W$, where c is an integer.

and $\sqrt{2} \in R$, then

$\sqrt{2}\,(a, b, c) = \left(\sqrt{2}a, \sqrt{2}b, \sqrt{2}c\right) \notin W$ [$\because$ *c is an integer but $\sqrt{2}$ c is not an integer*]

Thus W is not closed for scalar multiplication.

Hence W is not a vector space of V.

(*iii*) Let $(a, b, c) \in W$ and $\sqrt{2} \in \mathbf{R}$, then

$\sqrt{2}\,(a, b, c) = \left(\sqrt{2}a, \sqrt{2}b, \sqrt{2}c\right) \notin W$. $\left[\because \sqrt{2}a, \sqrt{2}b, \sqrt{2}c \notin Q\right]$

Thus V is not closed for scalar multiplication.

Hence W is not a sub-space of V.

(*iv*) Let $(a, b, c) \in W$, where $a \leq b \leq c$ and $-1 \in \mathbf{R}$, then

$(-1)(a, b, c) = (-a, -b, -c) \notin W$ $[\because -a > -b > -c]$

Thus W is not closed for scalar multiplication.

Hence W is not a sub-space of V.

(*v*) Let $(a, b, c) \in W$, where $a^2 + b^2 + c^2 \leq 1$ and $-2 \in \mathbf{R}$, then

$(-2)(a, b, c) = (-2a, -2b, -2c) \notin W$ $[\because 4(a^2 + b^2 + c^2)$ *may not be* $\leq 1]$

Thus W is not closed for scalar multiplication.

Hence W is not a sub-space of V.

(*vi*) Since $a - 3b + 4c = 0$, $\therefore\ a = 3b - 4c$.

Let us select two elements of W so as to satisfy the above condition as :

$x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$

i.e., $x = (3b_1 - 4c_1, b_1, c_1)$ and $y = (3b_2 - 4c_2, b_2, c_2)$.

To prove. W is a sub-space.

$\alpha x + \beta y \in W$, where $\alpha, \beta \in \mathbf{R}$.

For $\alpha x + \beta y = \alpha(3b_1 - 4c_1, b_1, c_1) + \beta(3b_2 - 4c_2, b_2, c_2)$

$= \{3(\alpha b_1 + \beta b_2) - 4(\alpha c_1 + \beta c_2), \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2\} = (3B - 4C, B, C)$

$= (A, B, C) \in W,$

where $A = 3B - 4C$ or $A - 3B + 4C = 0,$

where $A, B, C \in \mathbf{R}$ because $\alpha b_1 + \beta b_2 \in \mathbf{R}$, $\alpha c_1 + \beta c_2 \in \mathbf{R}$.

Hence W is a sub-space of V.

(*vii*) Let $\alpha \in R$ and $x, y \in W$.

Then $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$

such that $b_1 + 4c_1 = 0$, $b_2 + 4c_2 = 0$.

Now $x - y = (a_1, b_1, c_1) - (a_2, b_2, c_2)$

such that $(b_1 - b_2) + 4(c_1 - c_2) = (b_1 + 4c_1) - (b_2 + 4c_2) = 0 - 0 = 0$

$\therefore$ $x - y \in W$

and $\alpha x = (\alpha a_1, \alpha b_1, \alpha c_1)$

such that $(\alpha b_1) + 4(\alpha c_1) = \alpha(b_1 + 4c_1) = \alpha(0) = 0$

$\therefore$ $\alpha x \in W$.

Hence W is a vector space of V.

(*viii*) Since $a - b + c = 0$ and $2a + 3b - c = 0$,

$\therefore$ $a = b - c$ and $2a + 3b - c = 0$.

Let us select two elements of W so as to satisfy the above condition as :

$x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$

$$\left.\begin{array}{lll} \text{such that} & a_1 = b_1 - c_1 \ ; & 2a_1 + 3b_1 - c_1 = 0 \\ \text{and} & a_2 = b_2 - c_2 \ ; & 2a_2 + 3b_2 - c_2 = 0 \end{array}\right\} \quad \ldots(1)$$

Now $x - y = (a_1, b_1, c_1) - (a_2, b_2, c_2) = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$

such that $a_1 - a_2 = (b_1 - c_1) - (b_2 - c_2)$ [*Using* (*1*)]

and $2(a_1 - a_2) + 3(b_1 - b_2) - (c_1 - c_2) = (2a_1 + 3b_1 - c_1) - (2a_2 + 3b_2 - c_2)$

$= 0 - 0$ [*Using* (*1*)]

$= 0$

$\therefore$ $x - y \in W$

and $\alpha x = \alpha(a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$

such that $\alpha a_1 = \alpha(b_1 - c_1) = \alpha b_1 - \alpha c_1$

and $2(\alpha a_1) + 3(\alpha b_1) - (\alpha c_1) = \alpha(2a_1 + 3b_1 - c_1) = \alpha(0) = 0$

$\therefore$ $\alpha x \in W$.

Hence W is a sub-space of V.

Example 8. *Let $V_3(R) = \{(a, b, c) \mid a, b, c \in R\}$ be a vector space over reals. Show that $W = \{(0, b, c) \mid 0, b, c \in R\}$ is a sub-space of $V_3(R)$.*

Sol. Let $\alpha, \beta \in$ R and $x, y \in$ W such that

$x = (0, b_1, c_1)$ and $y = (0, b_2, c_2) \in$ W, where $b_1, c_1, b_2, c_2 \in$ R.

Now $\alpha x + \beta y = \alpha(0, b_1, c_1) + \beta(0, b_2, c_2) = (0, \alpha b_1, \alpha c_1) + (0, \beta b_2, \beta c_2)$

$$= (0, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

$$\in \text{W} \qquad [\because \alpha, \beta, b_1, b_2, c_1, c_2 \in R \Rightarrow \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2 \in R]$$

Thus $\alpha x + \beta y \in$ W $\forall \alpha, \beta \in$ R and $x, y \in$ W.

Hence W is a sub-space of V_3(R).

Example 9. *Let a, b, c be fixed elements of a field F. Show that*

$$W = \{(x, y, z) \mid ax + by + cz = 0 ; x, y, z \in F\}$$

is a sub-space of $V_3(F)$.

Sol. Since $(0, 0, 0) \in$ W $\qquad [\because a.0 + b.0 + c.0 = 0 ; 0 \in F]$

$\Rightarrow$ W is non-empty.

Let $\alpha, \beta \in$ F and $u, v \in$ W.

Then $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$,

where $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$ ...(1)

Now $\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2)$

$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

and $a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2) = a\alpha x_1 + a\beta x_2 + b\alpha y_1 + b\beta y_2 + c\alpha z_1 + c\beta z_2$

$$= \alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2)$$

$$= \alpha(0) + \beta(0) = 0$$

$\therefore$ $\alpha u + \beta v \in$ W.

Hence W is a sub-space of V_3 (F).

Example 10. (*i*) *Find whether $W = \{(a, b, a, b) \mid a, b \in Z\}$ is a sub-space of R^4 (R), where $R^4 = \{(a, b, c, d) \mid a, b, c, d \in R\}$.*

(*ii*) *Find whether $W \{(x, x, x, x) \mid x \in R\}$ is a sub-space of R^4 (R), where $R^4 = \{(x, y, z, w) \mid x, y, z, w \in R\}$.*

Sol. (*i*) Let $x = (a, b, a, b), y = (u, v, u, v)$ and $\alpha, \beta \in$ R and $a, b, u, v \in$ Z.

Then $\alpha x + \beta y = \alpha(a, b, a, b) + \beta(u, v, u, v) = (\alpha a, \alpha b, \alpha a, \alpha b) + (\beta u, \beta v, \beta u, \beta v)$

$$= (\alpha a + \beta u, \alpha b + \beta v, \alpha a + \beta u, \alpha a + \beta v)$$

$$= (\text{A, B, A, B}).$$

But A $= \alpha a + \beta u$ may not $\in$ Z when $a, u \in$ Z, but $\alpha, \beta \in$ R.

For Ex. If $a = 2$, $u = 3$ and $\alpha = \frac{1}{7}, \beta = \frac{2}{3}$, then

$$\alpha a + \beta u = \frac{1}{7} . 2 + \frac{2}{3} . 3 = \frac{16}{7} \notin Z$$

$\Rightarrow$ $\alpha a + \beta u \notin$ W.

Hence W is not a sub-space.

(*ii*) $W = \{(x, x, x, x) \mid x \in R\}$.

Let α, β be any two elements of W and $a, b \in R$.

Then $a\alpha + b\beta = a(x, x, x, x) + b(y, y, y, y) \mid y \in R$

$= (ax, ax, ax, ax) + (by, by, by, by) = (ax + by, ax + by, ax + by, ax + by)$

$= (A, A, A, A) \in W$ $[\because A = ax + by \in R \text{ as } a, b, x, y \in R]$

Hence W is a sub-space.

Example 11. *If a vector space is the set of real valued continuous functions over* **R**, *then show that set W of solutions of differential equation*

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

is a sub-space of V.

Sol. We have : $W = \left\{ y \mid 2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \right\}$,

where $y = f(x)$.

Since y satisfies the differential equation, therefore, it $\in$ W.

Now select $y_1, y_2 \in W$ so that

$$2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1 = 0 \text{ and } 2\frac{d^2y_2}{dx^2} - 9\frac{dy_2}{dx} + 2y_2 = 0.$$

In order to prove that W is a sub-space we have to show that

$\alpha y_1 + \beta y_2 \in W$, where $\alpha, \beta \in F$.

Now $2\frac{d^2}{dx^2}(\alpha y_1 + \beta y_2) - 9\frac{d}{dx}(\alpha y_1 + \beta y_2) + 2(\alpha y_1 + \beta y_2) = 0$

$$\Rightarrow \quad 2\alpha\frac{d^2y_1}{dx^2} + 2\beta\frac{d^2y_2}{dx^2} - 9\left(\alpha\frac{dy_1}{dx} + \beta\frac{dy_2}{dx}\right) + 2\alpha y_1 + 2\beta y_2 = 0$$

$$\Rightarrow \quad \alpha\left(2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1\right) + \beta\left(2\frac{d^2y_2}{dx^2} - 9\frac{dy_2}{dx} + 2y_2\right) = 0$$

$\Rightarrow \quad \alpha(0) + \beta(0) = 0$, which is true.

Since $\alpha y_1 + \beta y_2$ satisfies the given differential equation as and when y_1, y_2 satisfy it.

Hence W is a sub-space.

Example 12. *Which of the following set of vectors*

$x = (a_1, a_2 \ldots\ldots, a_n)$ *in* $\mathbf{R}^n$ *are sub-spaces of* $\mathbf{R}^n$? $(n \geq 3)$

(*i*) *all x such that* $a_1 \geq 0$

(*ii*) *all x such that* $a_1 + 3a_2 = a_3$

(*iii*) *all x such that* $a_2 = a_1^2$

(*iv*) *all x such that* $a_1 a_2 = 0$

(*v*) *all x such that* a_2 *is rational*

(*vi*) *all x such that* $x_1 + x_2 + \ldots\ldots + x_n = k$, *a fixed real.*

Sol. (*i*) Let W = $\{x : x \in \mathbf{R}^n$ and $a_1 \geq 0\}$.

Let $x = (a_1, a_2, \ldots\ldots, a_n), y = (b_1, b_2, \ldots\ldots, b_n) \in W$, where $a_1, b_1 \geq 0$.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n) \notin W$$

Then for $\alpha, \beta \in \mathbf{R}$, we have

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n) \notin W$$

Because if $a_1 = 3, b_1 = 3$ and $\alpha = -1, \beta = -2$,

Then $\alpha a_1 + \beta b_1 = -3 - 6 = -9 < 0$.

Hence W is not a sub-space of $\mathbf{R}^n$.

(*ii*) Let W = $\{x : x \in \mathbf{R}^n$ and $a_1 + 3a_2 = a_3\}$.

Let $x = (a_1, a_2, \ldots\ldots, a_n)$ and $y = (b_1, b_2, \ldots\ldots, b_n) \in W$, where $a_1 + 3a_2 = a_3$ and $b_1 + 3b_2 = b_3$.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n),$$

we have $(\alpha a_1 + \beta b_1) + 3(\alpha a_2 + \beta b_2) = \alpha(a_1 + 3a_2) + \beta(b_1 + 3b_2) = \alpha a_3 + \beta b_3$.

Thus $\alpha x + \beta y \in W$.

Hence W is a sub-space of $\mathbf{R}^n$.

(*iii*) Let W = $\{x \mid x \in \mathbf{R}^n$ and $a_2 = a_1^2\}$.

Let $x = (a_1, a_2, \ldots\ldots, a_n)$ and $y = (b_1, b_2, \ldots\ldots, b_n) \in W$, where $a_2 = a_1^2$ and $b_2 = b_1^2$.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n),$$

we have $(\alpha a_1 + \beta b_1)^2 \neq \alpha a_2 + \beta b_2$

[$\because$ *if* $a_2 = 9,\ a_1 = 3,\ b_2 = 16,\ b_1 = 4,\ \alpha = 3,\ \beta = 4,$ *we have* $(\alpha a_1 + \beta b_1)^2 = (9 + 16)^2 = 625$ *while* $\alpha a_2 + \beta b_2 = 48 + 65 = 113$]

Thus $\alpha x + \beta y \notin W$.

Hence W is not a sub-space of $\mathbf{R}^n$.

(*iv*) Let W = $\{x \mid x \in \mathbf{R}^n$ and $a_1 a_2 = 0\}$.

Let $x = (a_1, a_2, \ldots\ldots, a_n),\ y = (b_1, b_2, \ldots\ldots, b_n) \in W$, where $a_1 a_2 = 0,\ b_1 b_2 = 0$.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n), \text{ we have}$$

$$(\alpha a_1 + \beta b_1)(\alpha a_2 + \beta b_2) \neq 0 \quad \text{for } a_1 a_2 = 0,\ b_1 b_2 = 0.$$

Thus $\alpha x + \beta y \notin W$.

Hence W is not a sub-space of $\mathbf{R}^n$.

(v) Let W = { $x \mid x \in R^n$ and a_2 is rational}.

Let $x = (a_1, a_2, \ldots\ldots, a_n)$ and $y = (b_1, b_2 \ldots\ldots, b_n) \in W$,

where a_2, b_2 are rationals.

Now for $\alpha, \beta \in \mathbf{R}$, we have :

$$\alpha x + \beta y = (\alpha a_1 + \beta b_1, \ldots\ldots, \alpha a_n + \beta b_n).$$

Here $\alpha a_2 + \beta b_2$ is not always rational. [$\because$ *When* $a_2 = 4$, $b_2 = 5$, $\alpha = \sqrt{5}$, $\beta = 7$, *we have* $\alpha a_2 + \beta b_2 = 4\sqrt{5} + 35$, *which is not rational*]

Thus $\alpha x + \beta y \notin W$.

Hence W is not a sub-space of $\mathbf{R}^n$.

(vi) Let W = {$x \mid x = (x_1, x_2, \ldots\ldots, x_n) \in R$ and $x_1 + x_2 + \ldots\ldots + x_n = k$, a fixed real}

Case I. When $k \neq 0$.

Let $x = (1, 2, 0, \ldots\ldots, 0)$, where $x_1 + x_2 + \ldots\ldots + x_n = 3 \neq 0$

and $y = (-2, -1, 0, \ldots\ldots, 0)$, where $y_1 + y_2 + \ldots\ldots + y_n = -3 \neq 0$.

Then $x + y = (1 + (-2), 2 + (-1), 0, \ldots\ldots, 0)$

$= (-1, 1, \ldots\ldots, 0)$, where $(-1) + 1 + 0 + \ldots\ldots + 0 = 0$

$\notin W$.

Case II. When $k = 0$.

Let $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$,

where $x_1 + x_2 + \ldots\ldots + x_n = 0$ and $y_1 + y_2 + \ldots\ldots + y_n = 0$.

Let $\alpha, \beta \in R$.

Then $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots\ldots, \alpha x_n + \beta y_n)$

$$= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + \ldots\ldots + (\alpha x_n + \beta y_n)$$
$$= \alpha(x_1 + x_2 + \ldots\ldots + x_n) + \beta(y_1 + y_2 + \ldots\ldots + y_n)$$
$$= \alpha(0) + \beta(0) = 0.$$

Thus $\alpha x + \beta y \in W$; $\alpha, \beta \in R$.

Hence W is a sub-space of R.

Example 13. *Let V be a vector space of a function* $F \mid V \to R$. *Which of the following are sub-spaces ? In each case* $f(x) \in V$.

(i) *all f such that* $f(x^2) = [f(x)]^2$

(ii) *all f such that* $f(0) = f(1)$

(iii) *all f such that* $f(-1) = 0$

(iv) *all f such that* $f(3) = 1 + f(-5)$

(v) *all f which are continuous.*

Sol. (i) Let W = {$f \mid f \in V$ and $f(x^2) = [f(x)]^2$}.

Let $f, g \in W$. Then $f(x^2) = [f(x)]^2$ and $g(x^2) = [g(x)]^2$.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$(\alpha f+\beta g)(x^2) = (\alpha f)(x^2)+(\beta g)(x^2) = \alpha f(x^2)+\beta g(x^2)$$

and $$(\alpha f+\beta g)(x^2) = [\alpha f(x)+\beta g(x)]^2 = \alpha^2[f(x)]^2+\beta^2[g(x)]^2+2\alpha\beta f(x)\, g(x)$$

Thus $(\alpha f+\beta g)(x^2) \neq [(\alpha f+\beta g)(x)]^2$

$\Rightarrow \quad (\alpha f+\beta g) \notin W$.

Hence W is not a sub-space of V.

(*ii*) Let $W = \{f \mid f \in V \text{ and } f(0) = f(1)\}$

Let $f, g \in W$. Then $f(0) = f(1)$ and $g(0) = g(1)$ ...(1)

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$(\alpha f+\beta g)(0) = \alpha f(0)+\beta g(0) = \alpha f(1)+\beta g(1) \quad [\textit{Using (1)}]$$
$$= (\alpha f+\beta g)(1)$$

$\Rightarrow \quad (\alpha f+\beta g) \in W$.

Hence W is a sub-space of V.

(*iii*) Let $W = \{f \mid f \in V \text{ and } f(-1) = 0\}$.

Let $f, g \in W$. Then $f(-1) = 0$ and $g(-1) = 0$ ...(1)

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$(\alpha f+\beta g)(-1) = \alpha f(-1)+\beta g(-1) = \alpha . 0+\beta . 0 \quad [\textit{Using (1)}]$$
$$= 0$$

$\Rightarrow \quad \alpha f+\beta g \in W$.

Hence W is a sub-space of V.

(*iv*) Let $W = \{f \mid f \in V \text{ and } f(3) = 1+f(-5)\}$.

Let $f, g \in W$. Then $f(3) = 1+f(-5)$ and $g(3) = 1+g(-5)$ ...(1)

Now for $\alpha, \beta \in \mathbf{R}$, we have

$$(\alpha f+\beta g)(3) = \alpha f(3)+\beta g(3) = \alpha(1+f(-5))+\beta(1+g(-5)) \quad [\textit{Using (1)}]$$
$$= (\alpha+\beta)+(\alpha f+\beta g)(-5) = 1+(\alpha f+\beta g)(-5)$$

Thus $\alpha f+\beta g \notin W$.

Hence W is not a sub-space of V.

(*v*) Let $W = \{f \mid f \in V \text{ and } f \text{ is continuous}\}$.

Let $f, g \in W$. Then f and g are continuous.

Now for $\alpha, \beta \in \mathbf{R}$, we have

$\alpha f+\beta g$ is also continuous. [$\because$ *f, g are continuous*]

$\therefore \quad \alpha f+\beta g \in W$.

Hence W is a sub-space of V.

Example 14. *Prove that the set of all polynomials in one indeterminate x over a field F of degree less than or equal to n is a sub-space of the vector space of all polynomials over F.*

$$\{p(x) \mid p(x) = a_0+a_1x+a_2x^2+\ldots\},\ a_i \in F.$$

Sol. We know that P(x) is a vector space.

Let $W = \{p(x) \mid p(x) = a_0+a_1x+\ldots\ldots+a_nx^n\},\ a_i \in F$

i.e., W is a set of all polynomials of degree $\leq n$.

Clearly $W \subset P(x)$.

Let $p_1(x) = \sum_{i=0}^{n} a_i x^i$ and $p_2(x) = \sum_{i=0}^{n} b_i x^i \in W$,

then $ap_1(x) + bp_2(x)$, where $a, b \in F$

$$= a \sum_{i=0}^{n} a_i x^i + b \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (aa_i) x^i + \sum_{i=0}^{n} (bb_i) x^i = \sum_{i=0}^{n} (aa_i + bb_i) x^i,$$

which is again a polynomial of degree $\leq n$ and, therefore, $\in$ W.

Hence W is a sub-space.

Example 15. *Let V be a vector space of all 2×2 matrices over reals. Determine whether W is a sub-space of V or not, where :*

(*a*) *W consists of all matrices with non-zero determinant*

(*b*) *W consists of all matrices A s.t. $A^2 = A$.*

(*c*) $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},\ \textit{where } a, b \in R \right\}$

Sol. (*a*) Let $W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} ; x, y \in R \right\}$.

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$, W is a non-empty subset of V.

Consider $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$,

$$B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \in W \text{ and } \alpha, \beta \in R$$

$$\therefore\ \alpha A + \beta B = \alpha \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + \beta \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta x_2 & 0 \\ 0 & \alpha y_1 + \beta y_2 \end{bmatrix} \in W \qquad [\because \textit{Its det.} \neq 0]$$

Hence W is a sub-space of V.

(*b*) W is not a sub-space of V because W is not closed under addition.

To verify. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so that

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\therefore$ A $\in$ W.

But $\quad A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

and $(A+A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq A + A.$

Thus $A + A \notin W$.

(*c*) **Same as part (*a*).**

Example 16. *Let* $V = \{A \mid A = [a_{ij}]_{n \times n}, a_{ij} \in R\}$ *be a vector space over R. Show that* $W = \{A \in V \mid AX = XA \ \forall X \in V\}$ *is a sub-space of V(R).*

Sol. Since $OX = O = XO \ \forall \ X \in V$

$\Rightarrow \quad O \in W$

Thus W is non-empty.

Clearly $W \subset V$.

Let $\alpha, \beta \in R$ and $P, Q \in W$

$\Rightarrow \quad PX = XP$ and $QX = XQ \quad \forall X \in V$ ...(1)

$$\therefore \quad (\alpha P + \beta Q) X = (\alpha P)X + (\beta Q)X = \alpha (PX) + \beta (QX)$$
$$= \alpha (XP) + \beta (XQ) \quad [\text{Using } (1)]$$
$$= X (\alpha P) + X (\beta Q) = X (\alpha P + \beta Q)$$

$\Rightarrow \quad \alpha P + \beta Q \in W$.

Hence W is a vector space of V (R).

Example 17. *If V be the vector space of all square n × n matrices over reals. Examine whether the following are sub-spaces of V or not :*

(*i*) *Collection of all symmetric matrices*

(*ii*) *Collection of all skew-symmetric matrices*

(*iii*) *Collection of all scalar matrices*

(*iv*) *Collection of all singular matrices*

(*v*) *Collection of all diagonal matrices.*

Sol. We have

$$V = \{ A \mid A = [a_{ij}]_{n \times n} \ ; \ a_{ij} \in R\}$$

is a vector space over R.

(*i*) W = Collection of symmetric matrices

Clearly $W \subset V$.

$$\left.\begin{array}{ll} \text{Let } x, y \in W, \text{ where } & x = [b_{ij}] \text{ for which } b_{ij} = b_{ji} \\ \text{and} & y = [c_{ij}] \text{ for which } c_{ij} = c_{ji} \end{array}\right\} \quad ...(1)$$

Then $\alpha x + \beta y = \alpha [b_{ij}] + \beta [c_{ij}]$ for $\alpha, \beta \in \mathbf{R}$

$= [\alpha b_{ij} + \beta c_{ij}]$ [*By matrix addition and scalar multiplication*]

$= [d_{ij}]$, where $d_{ij} = \alpha b_{ij} + \beta c_{ij}$.

Now $d_{ji} = \alpha b_{ji} + \beta c_{ji} = \alpha b_{ij} + \beta c_{ij}$ [*Using (1)*]

$= d_{ij}$.

Thus $[d_{ij}]$ is also a symmetric matrix and, therefore, $\in$ W.

Hence W is a sub-space of V.

(*ii*) Here W = Collection of all skew-symmetric matrices

$$= \{[a_{ij}]_{n \times n} \mid a_{ij} = -a_{ji} \ \forall \ a_{ij} \in \mathbb{R}\}$$

Clearly W $\subset$ V.

$$\left.\begin{array}{llll} \text{Let } x, y \in W, \text{ where} & x = [a_{ij}] & \text{for which} & a_{ij} = -a_{ji} \\ \text{and} & y = [b_{ij}] & \text{for which} & b_{ij} = -b_{ji} \end{array}\right\} \quad \ldots(1)$$

Then $\alpha x + \beta y = \alpha [a_{ij}] + \beta [b_{ij}]$ for $\alpha, \beta \in \mathbb{R}$

$= [\alpha a_{ij} + \beta b_{ij}]$ [*By matrix addition and scalar multiplication*]

$= [c_{ij}]$, where $c_{ij} = \alpha a_{ij} + \beta b_{ij}$

Now $c_{ji} = \alpha a_{ji} + \beta b_{ji} = -\alpha a_{ij} - \beta b_{ij}$ [*Using (1)*]

$= -c_{ij}$

Thus $[c_{ij}]$ is also skew-symmetric and, therefore, $\in$ W.

Hence W is a sub-space of V.

(*iii*) Here W = Collection of all scalar matrices

$$= \{[a_{ij}]_{n \times n} \mid a_{ij} = k \text{ for } i = j, \text{ where } k \in \mathbb{R}$$
$$= 0 \text{ for } i \neq j\}$$

Clearly W $\subset$ V.

Now proceed exactly as in part (*a*).

(*iv*) Here W = Collection of all singular matrices

Clearly W is not a sub-space of V because W is not closed under addition.

$$\left[\text{For Ex.} \quad \text{Let A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \in W\right.$$

$$\left.\text{But } A + B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \notin W \text{ as } \det(A + B) = 6 \neq 0\right]$$

(*v*) Here W = Collection of all diagonal matrices

$$= \{[a_{ij}]_{n \times n} \mid a_{ij} = 0 \text{ for } i \neq j, \ a_{ij} \in \mathbb{R}\}$$

Clearly W $\subset$ V.

Now proceed exactly as in part (*a*)

4. Linear Combination

Def. *Let V be a vector space over F. Let $x_1, x_2, \ldots\ldots, x_n \in V$. Then any element 'x' which can be written of the type*

$$x = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n$$

$$= \sum_{i=1}^{n} a_i x_i \quad \text{for } a_i \in F, \qquad \text{where } 1 \leq i \leq n$$

is said to be **linear combination** *of the vectors $x_1, x_2, \ldots\ldots, x_n$ over F.*

Since V is a vector space, therefore, $x \in V$. *[By addition and scalar multiplication in V]*

Note. For $x_1, x_2, \ldots\ldots, x_n$, we get different linear combinations by taking different sets of scalars.

5. Generator of a vector space

Def. *Let S be a non-empty subset of a vector space V(F). Then S is said to be the generator of V(F) if each element of V can be expressed as a linear combination of the elements of S.*

Thus if S is the generator of the vector space V (F), and if $v \in V$, then there exist $v_1, v_2, \ldots\ldots, v_n \in S$ s. t.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n \quad \text{for } \alpha\text{'s} \in F, \text{ where } 1 \le i \le n.$$

For Ex. Let $V_2(R) = \{(\alpha, \beta) \mid \alpha, \beta \in R\}$ and

$S = \{(1, 0), (0, 1)\} = \{e_1, e_2\}$.

Let $v \in V$.

Then $v = (\alpha, \beta) = \alpha(1, 0) + \beta(0, 1) = \alpha e_1 + \beta e_2$

$\Rightarrow$ v is a linear combination of elements of S.

Hence S generates V.

SOLVED EXAMPLES

Example 1. (*i*) *Write the vector $x = (1, 7, -4)$ as a linear combination of vectors $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$ in vector space V_2* **(R)**.

(*ii*) *Write the vector $x = (2, -5, 4)$ as a linear combination of vectors $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$ in vector space V_3* **(R)**.

Sol. (*i*) Let $x = \alpha_1 x_1 + \alpha_2 x_2$, where $\alpha_1, \alpha_2 \in \mathbf{R}$

i.e., $(1, 7, -4) = \alpha_1(1, -3, 2) + \alpha_2(2, -1, 1) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$

$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$.

Comparing, $1 = \alpha_1 + 2\alpha_2$ ...(1)

$7 = -3\alpha_1 - \alpha_2$ ...(2)

and $-4 = 2\alpha_1 + \alpha_2$ ...(3)

Adding (2) and (3), $3 = -\alpha_1 \Rightarrow \alpha_1 = -3$.

Putting in (2), $7 = -3(-3) - \alpha_2$

$\Rightarrow$ $\alpha_2 = 9 - 7 = 2$.

[$\alpha_1 = -3$ and $\alpha_2 = 2$ verify (1) because $1 = -3 + 2(2)$ i.e., $1 = -3 + 4$]

$\therefore$ $x = -3x_1 + 2x_2$, which expresses x as a linear combination of x_1 and x_2.

(*ii*) Let $x = \alpha_1 x_1 + \alpha_2 x_2$, where $\alpha_1, \alpha_2 \in \mathbf{R}$

i.e., $(2, -5, 4) = \alpha_1(1, -3, 2) + \alpha_2(2, -1, 1) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$

$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$.

Comparing, $2 = \alpha_1 + 2\alpha_2$ ...(1)

$-5 = -3\alpha_1 - \alpha_2$ ...(2)

and $4 = 2\alpha_1 + \alpha_2$ ...(3)

Adding (2) and (3), $-1 = -\alpha_1 \Rightarrow \alpha_1 = 1$.

Putting in (3), $4 = 2(1) + \alpha_2 \Rightarrow \alpha_2 = 2$.

But $\alpha_1 = 1$ and $\alpha_2 = 2$ do not satisfy (1). $[\because\ 2 \neq 1 + 4]$

Hence x cannot be expressed as a linear combination of vectors x_1 and x_2.

Example 2. *For what value of k will the vector $x = (1, k, 5)$ in V_3 (**R**) is a linear combination of vectors $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$?*

Sol. Let $x = \alpha_1 x_1 + \alpha_2 x_2$, where $\alpha_1, \alpha_2 \in \mathbf{R}$

i.e., $$(1, k, 5) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -1, 1) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$$
$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2).$$

Comparing, $1 = \alpha_1 + 2\alpha_2$ ...(1)

$k = -3\alpha_1 - \alpha_2$ ...(2)

and $5 = 2\alpha_1 + \alpha_2$ ...(3)

(1) – 2(3) gives : $-9 = -3\alpha_1 \Rightarrow \alpha_1 = 3.$

Putting in (1), $1 = 3 + 2\alpha_2 \Rightarrow 2\alpha_2 = -2 \Rightarrow \alpha_2 = -1.$

Putting these in (2),

$$k = -3(3) - (-1) \Rightarrow k = -9 + 1.$$

Hence $k = -8.$

Example 3. *Write the vector $x = (1, -2, 5)$ as a linear combination of vectors.*

$$x_1 = (1, 1, 1,), \quad x_2 = (1, 2, 3), \quad x_3 = (2, -1, 1)$$

in vector space V_3 (**R**).

Sol. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

i.e., $$(1, -2, 5) = \alpha_1 (1, 1, 1) + \alpha_2 (1, 2, 3) + \alpha_3 (2, -1, 1)$$
$$= (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$$
$$= (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3).$$

Comparing, $1 = \alpha_1 + \alpha_2 + 2\alpha_3$ ...(1)

$-2 = \alpha_1 + 2\alpha_2 - \alpha_3$ ...(2)

and $5 = \alpha_1 + 3\alpha_2 + \alpha_3$ ...(3)

Adding (2) and (3), $3 = 2\alpha_1 + 5\alpha_2$ ...(4)

(1) + 2(2) gives : $-3 = 3\alpha_1 + 5\alpha_2$ ...(5)

(5) – (4) gives : $-6 = \alpha_1 \Rightarrow \alpha_1 = -6.$

Putting in (4), $3 = -12 + 5\alpha_2 \Rightarrow 5\alpha_2 = 15 \Rightarrow \alpha_2 = 3.$

Putting in (1), $1 = -6 + 3 + 2\alpha_3 \Rightarrow 2\alpha_3 = 4 \Rightarrow \alpha_3 = 2.$

Hence $x = -6x_1 + 3x_2 + 2x_3$, which expresses x as a linear combination of x_1, x_2 and x_3.

Example 4. *Is the vector $x = (2, -5, 3)$ in V_3 (**R**) a linear combination of the vectors*

$$x_1 = (1, -3, 2), \quad x_2 = (2, -4, -1), \quad x_3 = (1, -5, 7)?$$

Sol. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

i.e., $$(2, -5, 3) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -4, -1) + \alpha_3 (1, -5, 7)$$
$$= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3)$$
$$= (\alpha_1 + 2\alpha_2 + \alpha_3, -3\alpha_1 - 4\alpha_2 - 5\alpha_3, 2\alpha_1 - \alpha_2 + 7\alpha_3)$$

Comparing, $2 = \alpha_1 + 2\alpha_2 + \alpha_3$ ...(1)

$-5 = -3\alpha_1 - 4\alpha_2 - 5\alpha_3$ ...(2)

and $3 = 2\alpha_1 - \alpha_2 + 7\alpha_3$ ...(3)

5(1) + (2) gives : $5 = 2\alpha_1 + 6\alpha_2$ ...(4)

7(2) + 5(3) gives :

$$-20 = -11\alpha_1 - 33\alpha_2$$ ...(5)

(4) $\Rightarrow \alpha_1 + 3\alpha_2 = \dfrac{5}{2}$ ...(4′)

(5) $\Rightarrow \alpha_1 + 3\alpha_2 = \dfrac{20}{11}$ ...(5′)

These (4′) and (5′) are inconsistent equations.

Thus α_1 and α_2 and hence α_3 cannot be found.

Hence x is not a linear combination of x_1, x_2 and x_3.

Example 5. *Write the polynomial $V = t^2 + 4t - 3$ over $\mathbf{R}$ as a linear combination of the polynomials :*

$$e_1 = t^2 - 2t + 5,\quad e_2 = 2t^2 - 3t \text{ and } e_3 = t + 3.$$ (Pbi. U. 1997)

Sol. Let $V = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

i.e.,
$$t^2 + 4t - 3 = \alpha_1 (t^2 - 2t + 5) + \alpha_2 (2t^2 - 3t) + \alpha_3 (t + 3)$$
$$= (\alpha_1 + 2\alpha_2)t^2 + (-2\alpha_1 - 3\alpha_2 + \alpha_3)t + (5\alpha_1 + 3\alpha_3)$$

Comparing co-effs. of t^2,

$$1 = \alpha_1 + 2\alpha_2 \quad \text{...(1)}$$

Comparing co-effs. of t,

$$4 = -2\alpha_1 - 3\alpha_2 + \alpha_3 \quad \text{...(2)}$$

Comparing constants,

$$-3 = 5\alpha_1 + 3\alpha_3 \quad \text{...(3)}$$

3(2) – (3) gives : $15 = -11\alpha_1 - 9\alpha_2 \quad \Rightarrow \quad 11\alpha_1 + 9\alpha_2 = -15$ ...(4)

Multiplying (1) by 11, $11\alpha_1 + 22\alpha_2 = 11$ ...(5)

(5) – (4) gives : $13\alpha_2 = 26 \quad \Rightarrow \quad \alpha_2 = 2.$

Putting in (1), $1 = \alpha_1 + 4 \quad \Rightarrow \quad \alpha_1 = -3.$

Putting in (3), $-3 = -15 + 3\alpha_3 \quad \Rightarrow \quad 3\alpha_3 = 12 \quad \Rightarrow \quad \alpha_3 = 4.$

Hence $V = -3e_1 + 2e_2 + 4e_3.$

Example 6. *Write the vector $x = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in vector space of 2×2 matrices as a linear combination of*

$$x_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},\quad x_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},\quad x_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Sol. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, where $\alpha_1, \alpha_2, \alpha_3 \in F$

i.e.,
$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}$$

Comparing,

$$\left.\begin{array}{lcll} 3 = \alpha_1+\alpha_2+\alpha_3 & \Rightarrow & \alpha_1+\alpha_2+\alpha_3 = 3 & \ldots(1) \\ -1 = \alpha_1+\alpha_2-\alpha_3 & \Rightarrow & \alpha_1+\alpha_2-\alpha_3 = -1 & \ldots(2) \\ 1 = -\alpha_2 & \Rightarrow & \alpha_2 = -1 & \ldots(3) \\ -2 = -\alpha_1 & \Rightarrow & \alpha_2 = -1 & \ldots(4) \end{array}\right\} \ldots(A)$$

By (4) and (3), we have $\alpha_1 = 2, \qquad \alpha_2 = -1.$

Putting in (1), $2-1+\alpha_3 = 3 \Rightarrow \alpha_3 = 2.$

These satisfy (2) because $2-1-2=-1$ *i.e.*, $-1=-1$, which is true.

Thus the system (A) of equations is a consistent solution.

$\therefore \qquad x = 2x_1 - x_2 + 2x_3.$

Hence x is a linear combination of x_1, x_2 and x_3.

Example 7. *Write the vector* $x = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ *in vector space of* 2×2 *matrices, as a linear combination of*

$$x_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ x_3 = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}.$$

Sol. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, where $\alpha_1, \alpha_2, \alpha_3 \in F$

i.e., $$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ 0 & -\alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1+2\alpha_3 \\ \alpha_1+\alpha_2 & \alpha_2-\alpha_3 \end{bmatrix}$$

Comparing,

$$\left.\begin{array}{lcll} 3 = \alpha_1 & \Rightarrow & \alpha_1 = 3 & \ldots(1) \\ 1 = \alpha_1+2\alpha_3 & \Rightarrow & \alpha_1+2\alpha_3 = 1 & \ldots(2) \\ 1 = \alpha_1+\alpha_2 & \Rightarrow & \alpha_1+\alpha_2 = 1 & \ldots(3) \\ \text{and } -1 = \alpha_2-\alpha_3 & \Rightarrow & \alpha_2-\alpha_3 = -1 & \ldots(4) \end{array}\right\} \ldots(A)$$

From (1), $\alpha_1 = 3$

Putting in (3), $3+\alpha_2 = 1 \Rightarrow \alpha_2 = -2$

Putting in (2), $3+2\alpha_3 = 1 \Rightarrow 2\alpha_3 = -2 \Rightarrow \alpha_3 = -1$

These satisfy (4), $\qquad [\because -2+1=-1]$

Thus the system (A) of equations is a consistent solution.

Hence $\qquad x = 3x_1 - 2x_2 - x_3$.

Example 8. *Show that the vectors* $x_1 = (1, 2, 3)$, $x_2 = (0, 1, 2)$, $x_3 = (0, 0, 1)$ *generate* V_3 **(R)**.

Sol. Here we shall show that $x = (a, b, c) \in V_3(\mathbf{R})$ is a linear combination of x_1, x_2, x_3.

Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ are unknown scalars

i.e., $$(a, b, c) = \alpha_1(1, 2, 3) + \alpha_2(0, 1, 2) + \alpha_3(0, 0, 1)$$
$$= (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, \alpha_2, 2\alpha_2) + (0, 0, \alpha_3) = (\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2+\alpha_3)$$

Comparing, $a = \alpha_1 \Rightarrow \alpha_1 = a$ ...(1)

$b = 2\alpha_1 + \alpha_2 \Rightarrow 2\alpha_1 + \alpha_2 = b$ ...(2)

and $c = 3\alpha_1 + 2\alpha_2 + \alpha_3 \Rightarrow 3\alpha_1 + 2\alpha_2 + \alpha_3 = c$ ...(3)

From (1), $\alpha_1 = a.$

Putting in (2), $2a + \alpha_2 = b \Rightarrow \alpha_2 = b - 2a.$

Putting in (3), $3a + 2b - 4a + \alpha_3 = c \Rightarrow \alpha_3 = a - 2b + c.$

Thus $(a, b, c) = a(1, 2, 3) + (b - 2a)(0, 1, 2) + (a - 2b + c)(0, 0, 1).$

Hence x_1, x_2, x_3 generate V_3 **(R)**.

6. Linear Span

Def. *Let S be a non-empty subset of a vector space V(F). The set of all linear combinations of any finite number of elements of S is said to be* **linear span** *of S.*

(G.N.D.U. 1997 ; H.P.U. 1990)

This is denoted by L(S).

Thus $L(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in S \text{ and } \alpha_i \in F,\ 1 \le i \le n \right\}.$

7. Smallest Sub-space

Def. *Let V be a vector space over a field F and $S \subset V$. Then W, a sub-space of V(F), is said to be the smallest sub-space of V containing S iff*

(*i*) $S \subset W$ (*ii*) *W_1 is a sub-space of V(F) s.t. $S \subset W_1$, then $W \subset W_1$.*

The smallest sub-space is generally denoted by $< S >$.

Remark. For each subset S of V(F), there exists a unique smallest sub-space of V(F) containing S.

THEOREMS

Theorem I. *The linear span L(S) of any subset S of a vector space V(F) is a sub-space of V(F).*

(G.N.D.U. 1997, 90 ; P.U. 1992 ; H.P.U. 1990 ; Pbi. U. 1980)

Proof. Let $x, y \in L(S)$.

Then by def., $x = \sum \alpha_i x_i$, where $\alpha_i \in F$, $x_i \in S$; $i = 1, 2, \ldots, n$

and $y = \sum \beta_i y_i$, where $\beta_i \in F$, $y_i \in S$; $i = 1, 2, \ldots, m$.

To prove. *L(S) is a sub-space.*

In order to prove this, we are to prove that

for $\alpha, \beta \in F$ and $x, y \in L(S) \Rightarrow \alpha x + \beta y \in L(S)$.

Now $\alpha x + \beta y = \alpha \left(\sum_{i=1}^{n} \alpha_i x_i \right) + \beta \left(\sum_{j=1}^{m} \beta_j y_j \right) = \sum_{i=1}^{n} \alpha (\alpha_i x_i) + \sum_{j=1}^{m} \beta (\beta_j y_j)$

$= \sum_{i=1}^{n} (\alpha \alpha_j) x_i + \sum_{j=1}^{m} (\beta \beta_j) y_j$ [*Using Associative Law*]

$= (\alpha\alpha_1) x_1 + (\alpha\alpha_2) x_2 + \ldots\ldots + (\alpha\alpha_n) x_n + (\beta\beta_1) y_1 + (\beta\beta_2) y_2 + \ldots\ldots + (\beta\beta_m) y_m$

[$\because \alpha, \alpha_i \in F \Rightarrow \alpha\alpha_i \in F; \beta, \beta_j \in F \Rightarrow \beta\beta_j \in F;\ i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$]

$\Rightarrow$ $\alpha x + \beta y$ is expressed as a linear combination of finite number of vectors *viz.*

$x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$

of S and consequently it belongs to L(S).

Thus $\alpha x + \beta y \in L(S)$, where $\alpha, \beta \in F$ and $x, y \in L(S)$.
Hence L(S) is a sub-space of V(F).

Remark. If W is any other sub-space of V(F) containing S, then L (W) $\subset$ W.
L(S) is the smallest sub-space of V(F) containing S and is called sub-space **spanned** or **generated** by S.

Theorem II. *If S and T are any subsets of a vector space V(F), then*

(*i*) $S \subset L(T) \Rightarrow L(S) \subset L(T)$

(*ii*) $S \subset T \Rightarrow L(S) \subset L(T)$

(*iii*) *S is a sub-space of V(F)* $\Leftrightarrow L(S) = S$ (P.U. 1992 ; G.N.D.U. 1990 ; Pbi.U. 1986)

(*iv*) $L(S \cup T) = L(S) + L(T)$ (Pbi.U. 1997)

(*v*) $L(L(S)) = L(S)$.

Proof. (*i*) **Given :** $S \subset L(T)$.

Let $x \in L(S) \Rightarrow \exists\, x_1, x_2, \ldots\ldots, x_n \in S$;
$\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$
s.t. $x + \sum \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $x_i \in L(T)$ for $i = 1, 2, \ldots, n$ [$\because S \subset L(T)$]

$\Rightarrow$ $\sum \alpha_i x_i \in L(T)$ for $i = 1, 2, \ldots, n$ [$\because$ *L(T) is a sub-space of V(F)*]

$\Rightarrow$ $x \in L(T)$.

Thus $L(S) \subset L(T)$.
Hence $S \subset L(T) \Rightarrow L(S) \subset L(T)$.

(*ii*) **Given :** $S \subset T$.

Let $x \in L(S) \Rightarrow \exists\, x_1, x_2, \ldots, x_n \in S$;
$\alpha_1, \alpha_2, \ldots, \alpha_n \in F$
s.t. $x = \sum \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $x = \sum \alpha_i x_i \in L(T)$ [$\because S \subset T$ *so* $x_1, x_2, \ldots\ldots, x_n \in T$]

Thus $L(S) \subset L(T)$.
Hence $S \subset T \Rightarrow L(S) \subset L(T)$.

(*iii*) **Given :** S is a sub-space of V(F).
To prove . L(S) = S.

Let $x \in L(S) \Rightarrow \exists\, x_1, x_2, \ldots, x_n \in S$;
$\alpha_1, \alpha_2, \ldots, \alpha_n \in F$
s.t. $x = \sum \alpha_i x_i$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $x = \sum \alpha_i x_i \in S$

[$\because$ *S is a sub-space of V(F), hence it is closed for addition and scalar multiplication*]

$\therefore$ $L(S) \subset S$ ...(1)

Also obviously $S \subset L(S)$ ...(2)

From (1) and (2), L(S) = S.

Conversely : Given : L(S) = S.

To prove : S is a sub-space of V(F).

Since L(S) is a sub-space of V(F),

$\therefore$ S is also a sub-space of V(F).

(*iv*) Let $x \in L(S \cup T) \Rightarrow \exists\, x_1, x_2, \dots, x_n \in S \cup T$

and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$

s.t. $x = \sum \alpha_i x_i$ for $i = 1, 2, \dots, n$

$\Rightarrow$ $x = \sum \alpha_j x_j + \sum \alpha_k x_k$, where x_j's $\in$ S and remaining x_k's $\in$ T

[$\because$ *Each x_i is either an element of S or an element of T or an element of both, so dividing the elements x_i into elements belonging to S and belonging to T*]

$\Rightarrow$ $x \in L(S) + L(T)$

Thus $L(S \cup T) \subset L(S) + L(T)$ ...(1)

Let $z \in L(S) + L(T)$

$\Rightarrow$ $z = x + y$, where $x \in L(S)$, $y \in L(T)$

$\Rightarrow$ $z = \sum \alpha_j x_j + \sum \alpha_k x_k$, where x_j's $\in$ S, x_k's $\in$ T ;

α_j's, α_k's $\in$ F

$\Rightarrow$ $z = \sum \alpha_i x_i$, where x_i's $\in S \cup T$ [$\because \{x_i\} = \{x_j\} \cup \{x_k\}$]

$\Rightarrow$ $z \in L(S \cup T)$

$\Rightarrow$ $L(S) + L(T) \subset L(S \cup T)$ ...(2)

From (1) and (2), $L(S \cup T) = L(S) + L(T)$.

(*v*) Since L(S) is a sub-space of V(F),

$\therefore$ $L(L(S)) = L(S)$. [*Using (iii)*]

Theorem III. *Linear span of S i.e., L(S) is the smallest sub-space, where S is a subset of V(F).* (G.N.D.U. 1991 S, 90)

Proof. Let W be a sub-space of V such that $S \subset W$.

Let $v \in L(S)$ be an arbitrary element.

$\therefore$ $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_m v_m$, where $\alpha_i \in F, v_i \in S$; $i = 1, 2, \dots, m$

Clearly $S \subset W \Rightarrow v_1, v_2, \dots\dots, v_m \in W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_m v_m \in W$ [$\because$ *W is a sub-space of V(F)*]

$\Rightarrow v \in W$

$\Rightarrow L(S) \subset W$

$\Rightarrow$ L(S) is the smallest sub-space.

8. Linear Dependence and Linear Independence

(*i*) **Linear Dependent (L.D.). Def.** *If V is a vector space over field F, then the vectors $x_1, x_2, \dots, x_n \in V$ are said to be* **linearly dependent** *over F if $\exists$ elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ (not all zero) such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots\dots + \alpha_n x_n = 0.$$ (G.N.D.U. 1995 S)

For Example . *The vectors*

$x_1 = (1, 2, 3)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$ and $x_4 = (0, 0, 1)$ are linearly dependent (L.D.).

Sol. $1 \cdot x_1 + (-1)x_2 + (-2)x_3 + (-3)x_4$
$= 1(1, 2, 3) + (-1)(1, 0, 0) + (-2)(0, 1, 0) + (-3)(0, 0, 1)$
$= (1, 2, 3) + (-1, 0, 0) + (0, -2, 0) + (0, 0, -3) = (0, 0, 0) = \mathbf{0}.$

Thus $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \mathbf{0}$,
where $\alpha_1 = 1 \neq 0,\ \alpha_2 = -1 \neq 0,\ \alpha_3 = -2 \neq 0, \alpha_4 = -3 \neq 0.$

(*ii*) **Linear Independent (L.I.). Def.** *If V is a vector space over field F, then the vectors* $x_1, x_2, \ldots, x_n \in V$ *are said to be* **linearly independent** *if* $\exists$ *elements* $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ *(all zero) such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = \mathbf{0}.$$

For Example. *The vectors*

$x_1 = (1, 0, 0), x_2 = (0, 1, 0)$ *and* $x_3 = (0, 0, 1)$

are linearly independent (L.I.).

Sol. Let $\alpha_1, \alpha_2, \alpha_3$ be real numbers such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$$

$\Leftrightarrow \quad \alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1) = (0, 0, 0)$
$\Leftrightarrow \quad (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0, 0, 0)$
$\Leftrightarrow \quad (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$
$\Leftrightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$

9. Criteria for Linear Dependence and Linear Independence

(*i*) *An infinite sub-space S of a vector space is said to be L.I. if every subset of S is L.I.*

This is concluded from the definition of L.I. of a finite set.

(*ii*) *Any superset of L.D. set is L.D.*

Proof. Let S = $\{x_1, x_2, \ldots, x_m\}$ be a L.D. set

so that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0$ ...(1),

where all x_i' s are not zero.

Consider T = $\{x_1, x_2, \ldots, x_m, x\}$, which is clearly a superset of S.

Then $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m + 0x = \mathbf{0}$, [*Using (1)*]

where all α_i's are not zero.

Hence T is L.D.

(*iii*) *Any subset of a L.I. set is L.I.*

Proof. Let S = $\{x_1, x_2, \ldots, x_m\}$ be a L.I. set

so that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_m x_m = \mathbf{0}$, ...(1)

where $\alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_m = 0.$

Consider T = $\{x_1, x_2, \ldots\ldots, x_n\}$, where $1 \leq n \leq m$.

This is clearly a subset of S.

From (1), $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$

$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n + 0\ x_{n+1} + \ldots\ldots 0\, x_m = 0$

$\Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$ [$\because$ *S is L.I.*]

$\Rightarrow \quad$ T is L.I.

(iv) *Any subset of a vector space is either L.D. or L.I.*

(v) *A set containing only* **0** *vector i.e.,* $\{\mathbf{0}\}$ *is L.D.*

Proof. Let $S = \{\mathbf{0}\}$.

Since $\alpha \cdot \mathbf{0} = \mathbf{0}$ for any scalar α.

Hence S is L.D.

(vi) *A set containing the single non-zero vector is L.I.*

Proof. Let $S = \{x\}$, where $x \neq \mathbf{0}$.

Since $0 \cdot x = \mathbf{0} \Rightarrow \alpha = 0$ $[\because x \neq 0]$

$\therefore$ S is L.I.

(vii) *A set having one of the vectors as zero vector is L.D.*

Proof. Let $S = \{x_1, x_2, \ldots\ldots, x_n\}$ have one of the vectors ; say $x_i = \mathbf{0}$.

Then we have

$$0 \cdot x_1 + 0 \cdot x_2 + \ldots\ldots + 1 \cdot x_i + \ldots\ldots + 0 \cdot x_n = \mathbf{0}$$

which shows that the co-eff. of $x_i \neq 0$.

Hence S is L.D.

(viii) *If two vectors are L.D., then one of them is a scalar multiple of the other.*

Proof. Let x, y be two L.D. vectors of the vector space V (F), then there exist scalars $\alpha, \beta \in$ F (not both zero) such that

$$\alpha x + \beta y = \mathbf{0}.$$

If $\alpha \neq 0$, then $\alpha x = -\beta y \Rightarrow x = \left(-\frac{\beta}{\alpha}\right) y$

$\Rightarrow x$ is a scalar multiple of y.

Similar is the case when $\beta \neq 0$.

Hence the result.

THEOREMS

Theorem I. *If V (F) is a vector space, then the set S of non-zero vectors* $x_1, x_2, \ldots\ldots, x_n \in V$ *(i.e.,* $S = \{x_1, x_2, \ldots\ldots, x_n\} \subset V$*) is L.D. iff some elements of S is a linear combination of the others.*

Proof. Given. $S = \{x_1, x_2, \ldots\ldots, x_n\}$ is L.D. Then $\exists\ \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in$ F (not all zero), s.t.

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = \mathbf{0}$$

$\Rightarrow \sum_i \alpha_i x_i = \mathbf{0}$ [$\because$ *set S is L.D. so at least one of α_i's is not equal to zero, let* $\alpha_k \neq 0$]

$$\Rightarrow \sum_{i \neq k} \alpha_i x_i + \alpha_k x_k = \mathbf{0}$$

$\Rightarrow \alpha_k x_k = -\sum_{i \neq k} \alpha_i x_j$ [*By properties of vector space*]

$\Rightarrow x_k = \sum_{i \neq k} -\frac{\alpha_i}{\alpha_k} x_i$ [$\because \alpha_k \neq 0 \Rightarrow \alpha_k^{-1} \in F_j$, *thus by left inverse of* α_k]

This shows that some $x_k \in$ S is expressed as a linear combination of vectors $x_1, x_2, \ldots\ldots, x_{j-1}, x_{j+1}, \ldots\ldots, x_n \in$ S, *i.e.*,

$$x_j = \sum_{i \neq j} \beta_i x_i \text{ for } \beta_i\text{'s} \in F$$

$\Rightarrow$ $\sum_{i \neq j} \beta_i x_i + (-1) x_j = 0$ (*By postulates of vector space*)

Hence at least one co-efficient $-1 \neq 0$ showing that S is L.D.

Theorem II. *Let V be a vector space. Then*

(*a*) *The set* $\{v\}$ *is L.D. iff* $v = 0$.

(*b*) *The set* $\{v_1, v_2\}$ *is L.D. iff* v_1 and v_2 *are collinear i.e., iff one is a scalar multiple of the other.*

(*c*) *The set* $\{v_1, v_2, v_3\}$ *is L.D. iff* v_1, v_2 *and* v_3 *are coplanar i.e., iff one is a linear combination of the other two.* (G.N.D.U. 1985)

Proof. (*a*) Let $v = 0$.

$\therefore$ For each scalar $\alpha \neq 0$, $\alpha v = \alpha 0 = 0$.

$\therefore$ The set $\{v\}$ is L.D.

Conversely. Let $\{v\}$ be L.D.

$\therefore$ There exists a scalar $\alpha \neq 0$ s.t. $\alpha v = 0$

$\Rightarrow$ $v = 0$. $[\because \alpha \neq 0]$

(*b*) Let $\{v_1, v_2\}$ be L.D.

$\therefore$ There exist scalars α, β (with at least one of them ; say $\alpha \neq 0$)

s.t. $\alpha v_1 + \beta v_2 = 0$

$\Rightarrow$ $v_1 = -\left(\frac{\beta}{\alpha}\right) v_2$ $[\because \alpha \neq 0]$

$\Rightarrow$ v_1 is a scalar multiple of v_2

$\Rightarrow$ v_1 and v_2 are collinear.

Conversely. Let v_1 and v_2 be collinear.

$\therefore$ by def., v_1 is a scalar multiple of v_2

$\Rightarrow$ $v_1 = \alpha v_2$, where α is a scalar

$\Rightarrow$ $1 \cdot v_1 - \alpha v_2 = 0$

$\Rightarrow$ $\{v_1, v_2\}$ is L.D.

(*c*) Let $\{v_1, v_2, v_3\}$ be L.D.

$\therefore$ There exist scalars α, β, γ (with at least one of them ; say $\alpha \neq 0$) ;

s.t. $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$\Rightarrow$ $\alpha v_1 = (-\beta) v_2 + (-\gamma) v_3$

$\Rightarrow$ $v_1 = \left(-\frac{\beta}{\alpha}\right) v_2 + \left(-\frac{\gamma}{\alpha}\right) v_3$ $[\because \alpha \neq 0]$

$\Rightarrow$ v_1 is a linear combination of v_2 and v_3

$\Rightarrow$ v_1, v_2 and v_3 are coplanar.

Conversely. Let v_1, v_2 and v_3 be coplanar.

$\therefore$ One of them ; say v_1 is a linear combination of v_2 and v_3

$\therefore$ There exist scalars α_2, α_3 s.t.

$$v_1 = \alpha_2 v_2 + \alpha_3 v_3$$

$\Rightarrow$ $1 \cdot v_1 - \alpha_2 v_2 - \alpha_3 v_3 = 0$

$\therefore$ $\{v_1, v_2, v_3\}$ is L.D.

Theorem III. *If $V(F)$ is a vector space, then the set S of non-zero vectors $x_1, x_2, \ldots\ldots, x_n \in V$ (i.e., $S = \{x_1, x_2, \ldots\ldots, x_n\} \subset V$) is linearly dependent if and only if some vector $x_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceding vectors.* **(P.U. 1995)**

Proof. Given : $S = \{x_1, x_2, \ldots\ldots, x_n\}$ is L.D. Then $\exists$ scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ not all zero, s.t.

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$$

or $$\sum \alpha_i x_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Let k be the largest suffix of α (*i.e.*, the largest value of i) for which $\alpha_k \ne 0$. Then

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_k x_k + 0x_{k+1} + \ldots + 0x_n = 0$$

$$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_k x_k = 0 \qquad \ldots(1)$$

Suppose $k = 1$, then $\alpha_1 x_1 = 0$, but $\alpha_1 \ne 0$, so $x_1 = 0$,

which is contradictory because each x_i is a non-zero vector.

Hence $\qquad k > 1$, *i.e.*, $2 \le k \le n$.

By (1), we have $\alpha_k x_k = -\alpha_1 x_1 - \alpha_2 x_2 \ldots - \alpha_{k-1} x_{k-1}$.

As $\alpha_k \ne 0 \Rightarrow \alpha_k^{-1} \in F$. Thus by left inverse of α_k, we have

$$x_k = -\alpha_k^{-1}\alpha_1 x_1 - \alpha_k^{-1}\alpha_2 x_2 \ldots - \alpha_k^{-1}\alpha_{k-1} x_{k-1}$$
$$= \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{k-1} x_{k-1},$$

where $\beta_1 = -\alpha_k^{-1}\alpha_1$, $\beta_2 = -\alpha_k^{-1}\alpha_2, \ldots \beta_{k-1} = -\alpha_k^{-1}\alpha_{k-1} \in F$.

Thus x_k is expressed as a linear combination of its preceding vectors.

Conversely : Let some $x_i \in S$ be expressible as a linear combination of its preceding vectors *i.e.*

$$x_i = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{i-1} x_{i-1} \quad \text{for } \beta_i\text{'s} \in F$$

$$\Rightarrow \quad \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{i-1} x_{i-1} + (-1) x_i = 0 \qquad \textit{[By postulates of V]}$$

$$\Rightarrow \quad \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{i-1} x_{i-1} + (-1) x_i + 0x_{i+1} + \ldots + 0x_n = 0.$$

Thus at least one co-efficient $-1 \ne 0$ showing that S is L.D.

Note. If x is a linear combination of the set of vectors

$$x_1, x_2, \ldots\ldots, x_n,$$

then the vectors

$$\{x_1, x_2, \ldots\ldots, x_n\}$$

is linearly dependent.

SOLVED EXAMPLES

Example 1. *Fill in the blanks :*

(*i*) *Any set of vectors containing the zero vector as a member is linearly ......*

(*ii*) *Intersection of two linearly independent subsets of a vector space will be linearly ......*

(*iii*) *A system consisting of a simple non-zero vector is always linearly ......*

(*iv*) *In the vector space V_3* **(R)**, *the vectors (1, 0, 1) (2, 5, 0) and (– 1, 0, – 1) are linearly ......*

Sol. (*i*) A set of vectors containing the zero vector as a member is linearly *dependent*.

(*ii*) Intersection of two linearly independent subsets of a vector space will be linearly *independent*.

(*iii*) A system consisting of a simple non-zero vector is always linearly *independent*.

(*iv*) In the vector space V_3**(R)**, the vectors (1, 0, 1), (2, 5, 0) and (– 1, 0, – 1) are linearly *independent*.

Example 2. *In the vector space V_3* **(R)**, *let*

$$\alpha = (1, 2, 1), \; \beta = (3, 1, 5), \; \gamma = (3, -4, 7),$$

prove that the sub-spaces spanned by

$$S = \{\alpha, \beta\} \text{ and } T = \{\alpha, \beta, \gamma\} \text{ are same.}$$ **(P.U. 1996)**

Sol. L(T), the linear span T, is a set of vectors, which is a linear combination of vectors in T.

$\therefore$ $\quad L(T) = \{ a\alpha + b\beta + c\gamma \mid a, b, c \in R\}$

$= a(1, 2, 1) + b(3, 1, 5) + c(3, -4, 7)$ ...(1)

Let $(3, -4, 7) = x(1, 2, 1) + y(3, 1, 5)$ ...(2)

$\Rightarrow$ $(3, -4, 7) = (x, 2x, x) + (3y, y, 5y) = (x + 3y, \ 2x + y, \ x + 5y)$

$\Rightarrow$ $x + 3y = 3, 2x + y = -4$ and $x + 5y = 7$

Solving first two, $x = -3, \ y = 2$.

These satisfy third equation.

$\therefore$ From (2), $(3, -4, 7) = -3(1, 2, 1) + 2(3, 1, 5)$

$\Rightarrow$ $c(3, -4, 7) = -3c(1, 2, 1) + 2c(3, 1, 5)$

$\therefore$ $L(T) = a(1, 2, 1) + b(3, 1, 5) - 3c(1, 2, 1) + 2c(3, 1, 5)$ [*From (1)*]

$= (a - 3c)(1, 2, 1) + (b + 2c)(3, 1, 5)$

$= a'(1, 2, 1) + b'(3, 1, 5)$, where $a', b' \in R$

$= a'\alpha + b'\beta$

$= L(S)$, which is true.

Example 3. *Determine whether or not x and y are L.D.*

(*i*) $x = (4, 3, -2), \ y = (2, -6, 7)$

(*ii*) $x = (1, 2, 3, 4), \ y = (2, 4, 6, 8)$

(*iii*) $x = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{bmatrix}$

(*iv*) $x = t^3 + 3t + 4, \quad y = t^3 - 4t + 3.$

Sol. [**Remember :** Two vectors are L.D. iff one is a multiple of the other.]

(*i*) We have $x = (4, 3, -2)$ and $y = (2, -6, 7)$.

Since x cannot be expressed as a multiple of y,

$\therefore$ x, y are not L.D.

(*ii*) We have $x = (1, 2, 3, 4)$ and $y = (2, 4, 6, 8)$.

By Scalar Multiplication, $y = 2x$

Hence x, y are L.D.

(*iii*) We have $x = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{bmatrix}$.

Here $\quad y = \begin{bmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{bmatrix} = 2\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{bmatrix} = 2x.$

Hence x, y are L.D.

(*iv*) We have $x = t^3 + 3t + 4$ and $y = t^3 - 4t + 3$.

Since y cannot be expressed as a multiple of x.

$\therefore$ x, y are not L.D.

Example 4. *Determine whether the following system of vectors of* V_3 **(R)** *is linearly independent* :

$x_1 = (1, 2, -3), \ x_2 = (1, -3, 2), \ x_3 = (2, 1, -5).$ (P.U. 1985 S)

Sol. Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

$\therefore \quad \alpha_1(1, 2, -3) + \alpha_2(1, -3, 2) + \alpha_3(2, 1, -5) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 2\alpha_1, -3\alpha_1) + (\alpha_2, -3\alpha_2, 2\alpha_2) + (2\alpha_3, \alpha_3, -5\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + \alpha_2 + 2\alpha_3, 2\alpha_1 - 3\alpha_2 + \alpha_3, -3\alpha_1 + 2\alpha_2 - 5\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad \alpha_1 + \alpha_2 + 2\alpha_3 = 0 \quad ...(1)$

$2\alpha_1 - 3\alpha_2 + \alpha_3 = 0 \quad ...(2)$

and $\quad -3\alpha_1 + 2\alpha_2 - 5\alpha_3 = 0 \quad ...(3)$

(1) – 2 (2) gives : $-3\alpha_1 + 7\alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \frac{7}{3}\alpha_2$.

Putting in (1), $\frac{7}{3}\alpha_2 + \alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \quad \frac{10}{3}\alpha_2 + 2\alpha_3 = 0 \quad \Rightarrow \quad \alpha_3 = -\frac{5}{3}\alpha_2$.

$\therefore$ (1) and (2) $\Rightarrow \quad \alpha_1 = \frac{7}{3}\alpha_2, \ \alpha_3 = -\frac{5}{3}\alpha_2$.

These do not satisfy (3). $\left[\because \ -7\alpha_2 + 2\alpha_2 + \frac{25}{3}\alpha_2 \neq 0\right]$

Hence the given system is not L.D.

Example 5. *Prove that the following system of vectors of V_3 (**R**) are L.D. :*

(*i*) $x_1 = (1, 2, 3), \ x_2 = (4, 1, 5), \ x_3 = (-4, 6, 2)$ (G.N.D.U. 1997)

(*ii*) $x_1 = (1, 3, 2), \ x_2 = (1, -7, -8), \ x_3 = (2, 1, -1)$

(*iii*) $x_1 = (0, 2, -4), \ x_2 = (1, -2, -1), \ x_3 = (1, -4, 3)$

(*iv*) $x_1 = (1, 2, 3), \ x_2 = (1, 0, 0), \ x_3 = (0, 1, 0), \ x_4 = (0, 0, 1)$

(*v*) $x_1 = (1, 0, -1), \ x_2 = (2, 1, 3), \ x_3 = (-1, 0, 0), \ x_4 = (1, 0, 1)$

(*vi*) $x_1 = (3, 0, -3), \ x_2 = (-1, 1, 2), \ x_3 = (4, 2, -2), \ x_4 = (2, 1, 1)$.

Sol. (*i*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$.

$\therefore \quad \alpha_1(1, 2, 3) + \alpha_2(4, 1, 5) + \alpha_3(-4, 6, 2) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 2\alpha_1, 3\alpha_1) + (4\alpha_2, \alpha_2, 5\alpha_2) + (-4\alpha_3, 6\alpha_3, 2\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + 4\alpha_2 - 4\alpha_3, 2\alpha_1 + \alpha_2 + 6\alpha_3, 3\alpha_1 + 5\alpha_2 + 2\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad \alpha_1 + 4\alpha_2 - 4\alpha_3 = 0 \quad ...(1)$

$2\alpha_1 + \alpha_2 + 6\alpha_3 = 0 \quad ...(2)$

and $\quad 3\alpha_1 + 5\alpha_2 + 2\alpha_3 = 0 \quad ...(3)$

3(1) + 2(2) gives : $7\alpha_1 + 14\alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = -2\alpha_2$.

Putting in (1), $-2\alpha_2 + 4\alpha_2 - 4\alpha_3 = 0 \quad \Rightarrow \quad 2\alpha_2 = 4\alpha_3$

$\Rightarrow \quad \alpha_3 = \frac{1}{2}\alpha_2$.

$\therefore$ (1) and (2) $\Rightarrow \quad \alpha_1 = -2\alpha_2, \ \alpha_3 = \frac{1}{2}\alpha_2$.

These satisfy (3). $[\because \ -6\alpha_2 + 5\alpha_2 + \alpha_2 = 0]$

Hence the given system is L.D.

Aliter. Do upto equation (3) as above.

The equations (1), (2) and (3) can be put in the form AX = O

i.e., $$\begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where $$A = \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}.$$

Now $$\det A = \det \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 4 & -4 \\ 0 & -7 & 14 \\ 0 & -7 & 14 \end{bmatrix}$$

[*Operating* $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$]

$= 0$ [$\because$ R_2 *and* R_3 *are identical*]

$\Rightarrow$ $\alpha_1, \alpha_2, \alpha_3$ are not all zero.

Hence the given system is L.D.

(*ii*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1 \alpha_2, \alpha_3 \in \mathbf{R}$

$\therefore$ $\alpha_1 (1, 3, 2) + \alpha_2 (1, -7, -8) + \alpha_3 (2, 1, -1) = (0, 0, 0)$

$\Rightarrow$ $(\alpha_1, 3\alpha_1, 2\alpha_1) + (\alpha_2, -7\alpha_2, -8\alpha_2) + (2\alpha_3, \alpha_3, -\alpha_3) = (0, 0, 0)$

$\Rightarrow$ $(\alpha_1 + \alpha_2 + 2\alpha_3, 3\alpha_1 - 7\alpha_2 + \alpha_3, 2\alpha_1 - 8\alpha_2 - \alpha_3) = (0, 0, 0)$

$\Rightarrow$ $\alpha_1 + \alpha_2 + 2\alpha_3 = 0$ ...(1)

$3\alpha_1 - 7\alpha_2 + \alpha_3 = 0$ ...(2)

and $2\alpha_1 - 8\alpha_2 - \alpha_3 = 0$ ...(3)

Adding (2) and (3), $5\alpha_1 - 15\alpha_2 = 0$ [*Or multiply* (2) *by 2 and subtract* (1)]

$\Rightarrow$ $\alpha_1 = 3\alpha_2$.

If we choose $\alpha_2 = k$, then $\alpha_1 = 3k$ and putting in any, $\alpha_3 = -2k$.

Hence the given system is L.D.

(*iii*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

$\therefore$ $\alpha_1 (0, 2, -4) + \alpha_2 (1, -2, -1) + \alpha_3 (1, -4, 3) = (0, 0, 0)$

$\Rightarrow$ $(0, 2\alpha_1, -4\alpha_1) + (\alpha_2, -2\alpha_2, -\alpha_2) + (\alpha_3, -4\alpha_3, 3\alpha_3) = (0, 0, 0)$

$\Rightarrow$ $(\alpha_2 + \alpha_3, 2\alpha_1 - 2\alpha_2 - 4\alpha_3, -4\alpha_1 - \alpha_2 + 3\alpha_3) = (0, 0, 0)$

$\Rightarrow$ $\alpha_2 + \alpha_3 = 0$ ...(1)

$2\alpha_1 - 2\alpha_2 - 4\alpha_3 = 0 \Rightarrow \alpha_1 - \alpha_2 - 2\alpha_3 = 0$ ...(2)

and $-4\alpha_1 - \alpha_2 + 3\alpha_3 = 0$ ...(3)

Subtracting (3) from (2), $5\alpha_1 - 5\alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_3$.

From (1), $\alpha_2 = -\alpha_3$.

If we choose $\alpha_3 = k$, then $\alpha_1 = k$ and $\alpha_2 = -k$.

Hence the given system is L.D.

(*iv*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$.

$\therefore \quad \alpha_1 (1, 2, 3) + \alpha_2 (1, 0, 0) + \alpha_3 (0, 1, 0) + \alpha_4 (0, 0, 1) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, 0, 0) + (0, \alpha_3, 0) + (0, 0, \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + \alpha_2 + 0 + 0,\ 2\alpha_1 + 0 + \alpha_3 + 0,\ 3\alpha_1 + 0 + 0 + \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_3, 3\alpha_1 + \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad \alpha_1 + \alpha_2 = 0 \qquad \ldots(1)$

$2\alpha_1 + \alpha_3 = 0 \qquad \ldots(2)$

and $3\alpha_1 + \alpha_4 = 0 \qquad \ldots(3)$

$\therefore \quad \alpha_2 = -\alpha_1,\ \alpha_3 = -2\alpha_1$ and $\alpha_4 = -3\alpha_1$.

We can assign any value to α_1 and get the values of others.

Let $\alpha_1 = k$. Then $\alpha_2 = -k$, $\alpha_3 = -2k$ and $\alpha_4 = -3k$.

Thus all α's are not zero.

Hence the given system is L.D.

(*v*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$

$\therefore \quad \alpha_1 (1, 0, -1) + \alpha_2 (2, 1, 3) + \alpha_3 (-1, 0, 0) + \alpha_4 (1, 0, 1) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 0, -\alpha_1) + (2\alpha_2, \alpha_2, 3\alpha_2) + (-\alpha_3, 0, 0) + (\alpha_4, 0, \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4, \alpha_2, -\alpha_1 + 3\alpha_2 + \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad \alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 = 0 \qquad \ldots(1)$

$\alpha_2 = 0 \qquad \ldots(2)$

and $-\alpha_1 + 3\alpha_2 + \alpha_4 = 0 \qquad \ldots(3)$

From (2), $\alpha_2 = 0$.

From (3), $\alpha_1 = -\alpha_4$.

From (1), $-\alpha_4 + 0 - \alpha_3 + \alpha_4 = 0 \quad \Rightarrow \quad \alpha_3 = 0$.

Let $\alpha_4 = k$. Then $\alpha_1 = -k$, $\alpha_2 = 0$, $\alpha_3 = 0$ and $\alpha_4 = k$.

Thus all α's are not zero.

Hence the given system is L.D.

(*vi*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$

$\therefore \quad \alpha_1 (3, 0, -3) + \alpha_2 (-1, 1, 2) + \alpha_3 (4, 2, -2) + \alpha_4 (2, 1, 1) = (0, 0, 0)$

$\Rightarrow \quad (3\alpha_1, 0, -3\alpha_1) + (-\alpha_2, \alpha_2, 2\alpha_2) + (4\alpha_3, 2\alpha_3, -2\alpha_3) + (2\alpha_4, \alpha_4, \alpha_4) = (0, 0, 0)$

$\Rightarrow \quad (3\alpha_1 - \alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, -3\alpha_1 + 2\alpha_2 - 2\alpha_3 + \alpha_4) = (0, 0, 0)$

$$\Rightarrow \left.\begin{aligned} 3\alpha_1 - \alpha_2 + 4\alpha_3 + 2\alpha_4 &= 0 \\ \alpha_2 + 2\alpha_3 + \alpha_4 &= 0 \\ \text{and} \quad -3\alpha_1 + 2\alpha_2 - 2\alpha_3 + \alpha_4 &= 0 \end{aligned}\right\} \qquad \ldots(A)$$

These can be written as

$$\begin{bmatrix} 3 & -1 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ -3 & 2 & -2 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 3 & -1 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [Operating\ R_{31}(1)]$$

$$\Rightarrow \quad \begin{bmatrix} 3 & -1 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [Operating\ R_{32}(-1)]$$

The system (A) is equivalent to

$$3\alpha_1 - \alpha_2 + 4\alpha_3 + 2\alpha_4 = 0 \qquad ...(1)$$

$$\alpha_2 + 2\alpha_3 + \alpha_4 = 0 \qquad ...(2)$$

and $$2\alpha_4 = 0 \qquad ...(3)$$

(3) $\Rightarrow \alpha_4 = 0$.

Putting in (2), $\alpha_2 + 2\alpha_3 + 0 = 0 \quad \Rightarrow \quad \alpha_2 = -2\alpha_3$.

Putting in (1), $3\alpha_1 + 2\alpha_3 + 4\alpha_3 + 0 = 0 \quad \Rightarrow \quad 3\alpha_1 = -6\alpha_3$

$\Rightarrow \quad \alpha_1 = -2\alpha_3$.

Let us take $\alpha_3 = -k$.

Then $\alpha_1 = 2k$, $\alpha_2 = 2k$, $\alpha_3 = -k$, $\alpha_4 = 0$.

Hence the given system is L.D.

Example 6. *Let V be a vector space of real valued derivable functions on* $(0, \infty)$, *show that the set* $S = \{\sin x, \cos x, \sin(x + 1)\}$ *is L.D.* (G.N.D.U. 1992)

Sol. Please try yourself.

Example 7. *Prove that the following system of vectors of* V_3 **(R)** *are L.I.* :

(*i*) $x_1 = (1, 2, -3)$, $x_2 = (1, -3, 2)$, $x_3 = (2, -1, 5)$

(*ii*) $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$

(*iii*) $x_1 = (0, 1, -2)$, $x_2 = (1, -1, 1)$, $x_3 = (1, 2, 1)$.

Sol. (*i*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

$\therefore \quad \alpha_1(1, 2, -3) + \alpha_2(1, -3, 2) + \alpha_3(2, -1, 5) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 2\alpha_1, -3\alpha_1) + (\alpha_2, -3\alpha_2, 2\alpha_2) + (2\alpha_3, -\alpha_3, 5\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 + \alpha_2 + 2\alpha_3,\ 2\alpha_1 - 3\alpha_2 - \alpha_3,\ -3\alpha_1 + 2\alpha_2 + 5\alpha_3) = (0, 0, 0)$

$$\Rightarrow \quad \alpha_1 + \alpha_2 + 2\alpha_3 = 0 \qquad ...(1)$$

$$2\alpha_1 - 3\alpha_2 - \alpha_3 = 0 \qquad ...(2)$$

and $$-3\alpha_1 + 2\alpha_2 + 5\alpha_3 = 0 \qquad ...(3)$$

(1) + 2(2) gives : $5\alpha_1 - 5\alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2$ ...(4)

Putting in (1), $\alpha_2 + \alpha_2 + 2\alpha_3 = 0 \quad \Rightarrow \quad \alpha_3 = -\alpha_2$ ...(5)

Putting these in (3), $-3\alpha_2 + 2\alpha_2 - 5\alpha_2 = 0 \quad \Rightarrow \quad -6\alpha_2 = 0 \quad \Rightarrow \quad \alpha_2 = 0$.

(4) $\Rightarrow \alpha_1 = 0$ and (5) $\Rightarrow \alpha_3 = 0$.

Thus $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$.

Hence the given system is L.I.

Aliter. Do upto equation (3) as above.

The equations (1), (2) and (3) can be put in the form AX = O

i.e.,
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix}$.

Now $\det A = \det \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -5 \\ 0 & 5 & 11 \end{bmatrix}$

[*Operating* $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$]

$$= 1.\det \begin{bmatrix} -5 & -5 \\ 5 & 11 \end{bmatrix}$$

$$= 1.(-55+25) = -30 \neq 0$$

$\therefore\quad \alpha_1 = 0,\ \ \alpha_2 = 0,\ \ \alpha_3 = 0.$

Hence the given system is L.I.

(*ii*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

$\Rightarrow\quad \alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1) = (0, 0, 0)$

$\Rightarrow\quad (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0, 0, 0)$

$\Rightarrow\quad (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$

$\Rightarrow\quad \alpha_1 = 0,\ \ \alpha_2 = 0,\ \ \alpha_3 = 0.$

Hence the given system is L.I.

(*iii*) Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$

$\Rightarrow\quad \alpha_1 (0, 1, -2) + \alpha_2 (1, -1, 1) + \alpha_3 (1, 2, 1) = (0, 0, 0)$

$\Rightarrow\quad (0, \alpha_1, -2\alpha_1) + (\alpha_2, -\alpha_2, \alpha_2) + (\alpha_3, 2\alpha_3, \alpha_3) = (0, 0, 0)$

$\Rightarrow\quad (\alpha_2 + \alpha_3,\ \alpha_1 - \alpha_2 + 2\alpha_3, -2\alpha_1 + \alpha_2 + \alpha_3) = (0, 0, 0)$

$\Rightarrow\quad \alpha_2 + \alpha_3 = 0 \qquad \ldots(1)$

$\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \qquad \ldots(2)$

and $\quad -2\alpha_1 + \alpha_2 + \alpha_3 = 0 \qquad \ldots(3)$

(1) – (3) gives : $\quad 2\alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 = 0 \qquad \ldots(4)$

(1) + (2) gives : $\quad \alpha_1 + 3\alpha_3 = 0 \quad \Rightarrow \quad \alpha_3 = 0.$ [*Using* (4)]

Putting in (1), $\quad \alpha_2 + 0 = 0 \quad \Rightarrow \quad \alpha_2 = 0.$

Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0.$

Hence the given system is L.I.

Example 8. *Prove that the system of vectors*
$x_1 = (1, 2, 0), x_2 = (0, 3, 1),\ \ x_3 = (-1, 0, 1)$ *of* V_3 **(Q)** *is L.I. when Q is the field of rational numbers.*

Sol. Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{Q}$

$\Rightarrow\quad \alpha_1 (1, 2, 0) + \alpha_2 (0, 3, 1) + \alpha_3 (-1, 0, 1) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1, 2\alpha_1, 0) + (0, 3\alpha_2, \alpha_2) + (-\alpha_3, 0, \alpha_3) = (0, 0, 0)$

$\Rightarrow \quad (\alpha_1 - \alpha_3, 2\alpha_1 + 3\alpha_2, \alpha_2 + \alpha_3) = (0, 0, 0)$

$\Rightarrow \quad \alpha_1 - \alpha_3 = 0, \ 2\alpha_1 + 3\alpha_2 = 0, \ \alpha_2 + \alpha_3 = 0$

$\Rightarrow \quad \alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0.$

Hence the given system is L.I.

Example 9. *Prove that the vectors* $x_1 = (1 + i, 2i)$, $x_2 = (1, 1 + i)$ *in* V_2 **(C)** *are L.D. but in* V_2 **(R)** *are L.I.*

Sol. (*i*) Two vectors are linearly dependent if one is a multiple of the other.

Now if $x_1, x_2 \in V_2$ **(C)** and $1 + i \in$ **C**, we have :

$$(1 + i)x_2 = (1 + i)(1, 1 + i) = \left(1 + i, (1 + i)^2\right) = (1 + i, 1 + i^2 + 2i)$$
$$= (1 + i, 1 - 1 + 2i) = (1 + i, 2i)$$
$$= x_1$$

$\Rightarrow \quad x_1 = (1 + i)x_2$

$\Rightarrow \quad x_1, x_2$ are L.D.

(*ii*) If $x_1, x_2 \in V_2$ **(R)**, then x_1 cannot be a multiple of x_2. [$\because 1 + i \notin$ **R**]

Hence x_1, x_2 are L.I.

Example 10. *If* $v_1 = (2, -1, 0)$, $v_2 = (1, 2, 1)$ *and* $v_3 = (0, 2, -1)$, *show that* v_1, v_2, v_3 *are linearly independent. Express* $(3, 2, 1)$ *as a linear combination of* v_1, v_2, v_3. (Pbi. U. 1990, 86)

Sol. Let $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \mathbf{0}$, where $\alpha_1, \alpha_2, \alpha_3 \in$ **F**

$\therefore \quad \alpha_1(2, -1, 0) + \alpha_2(1, 2, 1) + \alpha_3(0, 2, -1) = (0, 0, 0)$

$\Rightarrow \quad (2\alpha_1, -\alpha_1, 0) + (\alpha_2, 2\alpha_2, \alpha_2) + (0, 2\alpha_3, -\alpha_3) = (0, 0, 0)$

$\Rightarrow \quad (2\alpha_1 + \alpha_2, -\alpha_1 + 2\alpha_2 + 2\alpha_3, \ \alpha_2 - \alpha_3) = (0, 0, 0)$

$\Rightarrow \quad 2\alpha_1 + \alpha_2 = 0 \qquad \ldots(1)$

$-\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \qquad \ldots(2)$

and $\quad \alpha_2 - \alpha_3 = 0 \qquad \ldots(3)$

These can be put in the form AX = O

i.e.,
$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$.

Now $\det A = \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$= \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ [*Operating* $C_2 \to C_2 + C_3$]

$= -\det \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = -(8 + 1) = -9 \neq 0$

$\therefore \quad \alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0.$

Hence the given system is L.I.

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{F}$

i.e., $(3, 2, 1) = \alpha_1 (2, -1, 0) + \alpha_2 (1, 2, 1) + \alpha_3 (0, 2, -1)$

$= (2\alpha_1, -\alpha_1, 0) + (\alpha_2, 2\alpha_2, \alpha_2) + (0, 2\alpha_3, -\alpha_3)$

$= (2\alpha_1 + \alpha_2, -\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 - \alpha_3)$

Comparing, $3 = 2\alpha_1 + \alpha_2$ ...(1)

$2 = -\alpha_1 + 2\alpha_2 + 2\alpha_3$ ...(2)

$1 = \alpha_2 - \alpha_3$ ...(3)

(2) + 2(3) gives : $4 = -\alpha_1 + 4\alpha_2$ ...(4)

(1) + 2(4) gives : $11 = 9\alpha_2 \quad \Rightarrow \quad \alpha_2 = \dfrac{11}{9}$.

Putting in (1), $3 = 2\alpha_1 + \dfrac{11}{9} \quad \Rightarrow 2\alpha_1 = \dfrac{16}{9} \quad \Rightarrow \alpha_1 = \dfrac{8}{9}$.

Putting in (3), $1 = \dfrac{11}{9} - \alpha_3 \quad \Rightarrow \quad \alpha_3 = \dfrac{2}{11}$.

Hence $(3, 2, 1) = \dfrac{8}{9}(2, -1, 0) + \dfrac{11}{9}(1, 2, 1) + \dfrac{2}{9}(0, 2, -1)$,

which is the reqd. L.C.

Example 11. *If x, y, z are L.I. vectors in a vector space $V(\mathbf{F})$, then prove that*

(*i*) $x + y,\ y + z,\ z + x$

(*ii*) $x + y,\ x - y,\ x - 2y + z$

are also L.I.

Sol. (*i*) Let $\alpha, \beta, \gamma \in \mathbf{F}$ such that

$\alpha (x + y) + \beta (y + z) + \gamma (z + x) = \mathbf{0}$ ...(1)

$\Rightarrow \quad (\alpha + \gamma) x + (\alpha + \beta) y + (\beta + \gamma) z = \mathbf{0}$ [*By postulates of V*]

But x, y, z are L.I. [*Given*]

$\therefore \quad \alpha + 0\beta + \gamma = 0$ ...(2)

$\alpha + \beta + 0\gamma = 0$ ...(3)

and $0\alpha + \beta + \gamma = 0$ ...(4)

These $\Rightarrow \quad \alpha = \beta = \gamma = 0$.

Hence (1) holds only if $\alpha = 0,\ \beta = 0,\ \gamma = 0$, which shows that the vectors $x + y,\ y + z,\ z + x$ are L.I.

(*ii*) Let $\alpha, \beta, \gamma \in \mathbf{F}$ such that

$\alpha (x + y) + \beta (x - y) + \gamma (x - 2y + z) = \mathbf{0}$ ...(1)

$\Rightarrow \quad (\alpha + \beta + \gamma) x + (\alpha - \beta - 2\gamma) y + \gamma z = \mathbf{0}$

But x, y, z are L.I. [*Given*]

$\therefore \quad \alpha + \beta + \gamma = 0$ ...(2)

$\alpha - \beta - 2\gamma = 0$ ...(3)

and $\gamma = 0$ ...(4)

These $\Rightarrow \quad \alpha = \beta = \gamma = 0$.

Hence (1) holds only if $\alpha = 0, \beta = 0, \gamma = 0$, which shows that the vectors $x + y, x - y, x - 2y + z$ are L.I.

Example 12. *In the vector space of polynomials of degree ≤ 4, which of the following sets are linearly independent :*

(*i*) $1+x,\ x+x^2,\ x^2+x^3,\ x^3+x^4,\ x^4-1$ (P.U. 1995)

(*ii*) $x^4-x,\ x^3+1,\ x^2-1,\ x$

(*iii*) $1,\ 1+x,\ (1+x)^2,\ (1+x)^3,\ (1+x)^4$.

Sol. (*i*) Here $x^4-1=(x^4+x^3)-(x^3+x^2)+(x^2+x)-(x+1)$

Thus x^4-1 is a linear combination of others.

Hence the set is L.D.

(*ii*) Here none can be expressed as a linear combination of others.

$\therefore$ The set is L.I.

(*iii*) **Please try yourself.** **[Ans.** L.I.]

Example 13. *Let V be a vector space of real valued derivable functions on $(0, \infty)$, then show that the set*

$$S=\{x^2 e^x,\ xe^x,\ (x^2+x-1)\,e^x\}$$

is L.I. (G.N.D.U. 1992 S)

Sol. We have $S=\{x^2e^x,\ xe^x,\ (x^2+x-1)\,e^x\}$.

Here none can be expressed as a linear combination of others.

Hence the set is L.I.

Example 14. *Prove that the four vectors :*

$$x_1=(1,0,0),\ \ x_2=(0,1,0),\ \ x_3=(0,0,1),\ \ x_4=(1,1,1)$$

in V_3 **(C)** *form a linearly dependent set, but any three of them are linearly independent.*

Sol. (*i*) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{C}$

such that $\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3+\alpha_4 x_4=0$ ...(1)

i.e., $\alpha_1(1,0,0)+\alpha_2(0,1,0)+\alpha_3(0,0,1)+\alpha_4(1,1,1)=(0,0,0)$

$\Rightarrow$ $(\alpha_1,0,0)+(0,\alpha_2,0)+(0,0,\alpha_3)+(\alpha_4,\alpha_4,\alpha_4)=(0,0,0)$

$\Rightarrow$ $(\alpha_1+\alpha_4,\ \alpha_2+\alpha_4,\ \alpha_3+\alpha_4)=(0,0,0)$

$\therefore$ $\alpha_1+\alpha_4=0 \Rightarrow \alpha_1=-\alpha_4$

$\alpha_2+\alpha_4=0 \Rightarrow \alpha_2=-\alpha_4$

$\alpha_3+\alpha_4=0 \Rightarrow \alpha_3=-\alpha_4$.

Thus if $\alpha_4=-k$, then $\alpha_1=k,\ \alpha_2=k,\ \alpha_3=k$, showing that

$$x_1+x_2+x_3-x_4=0. \qquad [\textit{Using } (1)]$$

Hence $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are L.D.

(*ii*) Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ such that $\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3=0$

$\Rightarrow$ $\alpha_1(1,0,0)+\alpha_2(0,1,0)+\alpha_3(1,1,1)=(0,0,0)$

$\Rightarrow$ $(\alpha_1,0,0)+(0,\alpha_2,0)+(\alpha_3,\alpha_3,\alpha_3)=(0,0,0)$

$\Rightarrow$ $(\alpha_1+\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_3)=(0,0,0)$

$\therefore$ $\alpha_1+\alpha_3=0$ ...(2)

$\alpha_2+\alpha_3=0$ ...(3)

and $\alpha_3=0$ ...(4)

Thus we have $\alpha_1=0,\ \alpha_2=0,\ \alpha_3=0$, which shows that the vectors x_1, x_2, x_4 are L.I.

Similarly we can prove for other collections of three vectors.

Example 15. *Under what condition on scalar $a_1 \in \mathbf{C}$ are the vectors $(1 + a_1, 1 - a_1)$ and $(1 - a_1, 1 + a_1)$ in $V_2(C)$ linearly dependent?*

Sol. Since the given vectors are L.D.,

$\therefore \quad (1 + a_1, 1 - a_1) = \alpha(1 - a_1, 1 + a_1)$ for $\alpha \in \mathbf{C}$

$\Rightarrow \quad (1 + a_1, 1 - a_1) = (\alpha(1 - a_1), \alpha(1 + a_1))$

$$\therefore \qquad 1 + a_1 = \alpha - \alpha a_1 \qquad \ldots(1)$$

and $$1 - a_1 = \alpha + \alpha a_1 \qquad \ldots(2)$$

By (1) and (2), for addition in F, we have

$$2 = 2\alpha \quad \Rightarrow \quad \alpha = 1.$$

Then by (2) for $\alpha = 1$, we have :

$$1 - a_1 = 1 + a_1 \quad \Rightarrow \quad 2a_1 = 0 \quad \Rightarrow \quad a_1 = 0.$$

Hence $a_1 = 0$ is the required condition.

Example 16. *Under what condition on the scalars $a_1, b_1 \in \mathbf{C}$ are the vectors $(1, a_1)$ and $(1, b_1)$ in $V_2(C)$ linearly dependent?*

Sol. Since the given vectors are L.D.,

$\therefore \quad (1, a_1) = \alpha(1, b_1)$ for $\alpha \in \mathbf{C}$

$\Rightarrow \quad (1, a_1) = (\alpha, \alpha b_1)$

$$\therefore \qquad 1 = \alpha \qquad \ldots(1)$$

and $$a_1 = \alpha b_1 \qquad \ldots(2)$$

From (1) and (2), $a_1 = b_1$, which is the required condition.

Example 17. *Show that vectors (a_1, a_2) and (b_1, b_2) in $V_2(C)$ are linearly independent iff $a_1 b_2 = a_2 b_1$.*

Sol. Given : (a_1, a_2) and (b_1, b_2) are L.D.

To prove : $a_1 b_2 = a_2 b_1$.

Since (a_1, a_2) and (b_1, b_2) are L.D.,

$\therefore \quad (b_1, b_2) = \alpha(a_1, a_2)$ for $\alpha \in \mathbf{C} = (\alpha a_1, \alpha a_2)$

i.e., $$b_1 = \alpha a_1 \qquad \ldots(1)$$

and $$b_2 = \alpha a_2 \qquad \ldots(2)$$

$$(1) \Rightarrow \quad \alpha = b_1 a_1^{-1} \text{ provided } a_1 \neq 0 \qquad \ldots(3)$$

$$(2) \Rightarrow \quad b_2 = (b_1 a_1^{-1}) a_2 \qquad [\textit{Using (3)}]$$

$$= a_2 (b_1 a_1^{-1})$$

$$\Rightarrow \quad a_1 b_2 = a_2 b_1.$$

If $a_1 = 0$, then $b_2 = \alpha 0 = 0$ and we have

$$b_2 0 = a_1 0 \quad \Rightarrow \quad b_2 a_1 = a_2 b_1 \quad \Rightarrow \quad a_1 b_2 = a_2 b_1.$$

Hence the result.

Conversely. Given : $a_1 b_2 = a_2 b_1$.

To prove : (a_1, a_2) and (b_1, b_2) are L.D.

Now $a_1 b_2 = a_2 b_1 \quad \Rightarrow \quad a_2 = a_1 b_2 b_1^{-1}$ provided $b_1 \neq 0$

$$\Rightarrow \quad a_2 = (a_1 b_1^{-1}) b_2 \qquad [\because \; a_1, a_2, b_1, b_2 \in \mathbf{F}]$$

$$\Rightarrow \quad a_2 = \alpha b_2 \qquad [\textit{where } \alpha = a_1 b_1^{-1}]$$

Also $\quad a_1 = \alpha b_1$ $\quad$ [*where* $a_1 = \alpha b_1$]

Thus $\quad (a_1, a_2) = \alpha (b_1, b_2)$.

When $b_1 = 0$. $\quad$ Then $a_2 0 = b_2 a_1 \Rightarrow 0 = b_2 a_1$

i.e., $\quad$ either $b_1 = 0$ $\quad$ or $\quad a_1 = 0$.

(*i*) *If* $b_1 = 0$, then one of the vectors is zero

$\Rightarrow$ vectors (a_1, a_2), (b_1, b_2) are L.D.

(*ii*) *If* $a_1 = 0$, then the vectors are $(0, b_2)$, $(0, a_2)$ and one is a multiple of the other such $(0, b_2) = b_2 a_2^{-1} (0, a_2)$ when $a_2 \neq 0$

$\Rightarrow$ the vectors are L.D.

Also if $a_2 = 0$, then we have one of the vectors as zero

$\Rightarrow$ the vectors are L.D.

Hence the result.

Example 18. *If V*(**R**) *be a vector space of* 2×3 *matrices over* **R**, *then show that the matrices*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}; \; B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}; \; C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

in V(**R**) *are linearly independent.*

Sol. Let α, β, γ be the scalars in **R** such that

$$\alpha \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + \gamma \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = O \qquad \text{(zero matrix)}$$

$$\Rightarrow \begin{bmatrix} 2\alpha + \beta + 4\gamma & \alpha + \beta - \gamma & -\alpha - 3\beta + 2\gamma \\ 3\alpha - 2\beta + \gamma & -2\alpha - 2\gamma & 4\alpha + 5\beta + 3\gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{...(1)}$$

Then $\quad 2\alpha + \beta + 4\gamma = 0, \; \alpha + \beta - \gamma = 0, \; -\alpha - 3\beta + 2\gamma = 0$

and $\quad 3\alpha - 2\beta + \gamma = 0, \; -2\alpha - 2\gamma = 0, \; 4\alpha + 5\beta + 3\gamma = 0.$

By second and fifth, we get :

$$2\alpha + \beta = 0.$$

Then by first, we get :

$$\gamma = 0$$

and so by fifth, we get

$$\alpha = 0, \; \beta = 0.$$

Thus (1) is true only if $\alpha = \beta = \gamma = 0$ and so the matrices A, B, C in V(**R**) are linearly independent.

Example 19. *Show that* (*i*) *row vectors,* (*ii*) *column vectors comprising the matrix*

$$\begin{bmatrix} 2 & 7 & 5 \\ 3 & -6 & 2 \\ 1 & 17 & 7 \end{bmatrix}$$

are linearly independent.

Sol. (*i*) The row vectors are :

[2, 7, 5], [3 −6 2] and [1 17 7].

Let α, β, γ be the scalars in **R** such that

$$\alpha\,[2, 7, 5] + \beta\,[3\ \ -6\ \ 2] + \gamma\,[1\ \ 17\ \ 7] = O$$

$$\Rightarrow \quad [2\alpha + 3\beta + \gamma,\ 7\alpha - 6\beta + 17\gamma,\ 5\alpha + 2\beta + 7\gamma] = [0\ \ 0\ \ 0] \qquad \ldots(1)$$

Then $2\alpha + 3\beta + \gamma = 0$ ...(2)

$7\alpha - 6\beta + 17\gamma = 0$ ...(3)

and $5\alpha + 2\beta + 7\gamma = 0$ ...(4)

(2), (3) and (4) $\Rightarrow$ $\alpha = \beta = \gamma = 0$.

Hence the row vectors are L.I.

(*ii*) The column vectors are

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -6 \\ 17 \end{bmatrix} \text{and} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}.$$

Let α, β, γ be the scalars in **R** such that

$$\alpha \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ -6 \\ 17 \end{bmatrix} + \gamma \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} = \mathbf{O}$$

$$\Rightarrow \quad \begin{bmatrix} 2\alpha + 7\beta + 5\gamma \\ 3\alpha - 6\beta + 2\gamma \\ \alpha + 17\beta + 7\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots(1)$$

Then $2\alpha + 7\beta + 5\gamma = 0$ ...(2)

$3\alpha - 6\beta + 2\gamma = 0$ ...(3)

and $\alpha + 17\beta + 7\gamma = 0$ ...(4)

(2), (3) and (4) $\Rightarrow$ $\alpha = \beta = \gamma = 0$.

Hence the column vectors are L.I.

Example 20. *Find a if the vectors*

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} \textit{are L.D.}$$ (G.N.D.U. 1986)

Sol. Let α, β, γ be scalar in **R** such that

$$\alpha \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \gamma \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \quad \begin{bmatrix} \alpha + \beta + a\gamma \\ -\alpha + 2\beta \\ 3\alpha - 3\beta + \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then $\alpha + \beta + a\gamma = 0$ ...(1)

$-\alpha + 2\beta = 0$ ...(2)

and $3\alpha - 3\beta + \gamma = 0$ ...(3)

From (2), $\alpha = 2\beta$ ...(4)

Putting in (3), $6\beta - 3\beta + \gamma = 0 \Rightarrow 3\beta + \gamma = 0$

$\Rightarrow \quad \beta = -\dfrac{\gamma}{3}.$

Putting in (4), $\alpha = -\dfrac{2\gamma}{3}.$

Putting in (1), $-\dfrac{2\gamma}{3} - \dfrac{\gamma}{3} + a\gamma = 0 \Rightarrow a\gamma - \gamma = 0$

$\Rightarrow \quad a - 1 = 0$ [$\because$ *$\gamma \neq 0$, otherwise if $\gamma = 0$, then $\alpha = \beta = 0$ also and then the system is L.I.*]

Hence $a = 1$.

Example 21. *If $x_1 = (1, 2, -1)$, $x_2 = (2, -3, 2)$, $x_3 = (4, 1, 3)$ and $x_4 = (-3, 1, 2)$ be vectors in V_3 (**R**), show that*

$$L(\{x_1, x_2\}) \neq L(\{x_3, x_4\}).$$

Sol. If possible, let us assume that $L(\{x_1, x_2\}) = L(\{x_3, x_4\})$.

This means that there exist scalars $\alpha_1, \alpha_2 \in \mathbf{R}$ for arbitrary $a_1, a_2 \in \mathbf{R}$

s.t. $\alpha_1 x_1 + \alpha_2 x_2 = a_1 x_3 + a_2 x_4$

i.e., $\alpha_1 (1, 2, -1) + \alpha_2 (2, -3, 2) = a_1 (4, 1, 3) + a_2 (-3, 1, 2)$

$\Rightarrow (\alpha_1 + 2\alpha_2, 2\alpha_1 - 3\alpha_2, -\alpha_1 + 2\alpha_2) = (4a_1 - 3a_2, a_1 + a_2, 3a_1 + 2a_2)$

$\Rightarrow \quad \alpha_1 + 2\alpha_2 = 4a_1 - 3a_2$ ...(1)

$2\alpha_1 - 3\alpha_2 = a_1 + a_2$ ...(2)

and $-\alpha_1 + 2\alpha_2 = 3a_1 + 2a_2$ ...(3)

Solving (1) and (3), $\alpha_1 = \dfrac{1}{2}(a_1 - 5a_2)$ and $\alpha_2 = \dfrac{1}{4}(7a_1 - a_2)$.

Putting in (2), $2\alpha_1 - 3\alpha_2 = 2\left[\dfrac{1}{2}(a_1 - 5a_2)\right] - 3 \cdot \dfrac{1}{4}(7a_1 - a_2)$

$$= (a_1 - 5a_2) - \frac{3}{4}(7a_1 - a_2) = -\frac{17}{4}a_1 - \frac{17}{4}a_2 \neq a_1 + a_2$$

$\therefore$ (2) is not satisfied.

Hence $L(\{x_1, x_2\}) \neq L(\{x_3, x_4\})$.

Example 22. *Prove that the set of vectors $\{x_1, x_2, \ldots\ldots, x_n\}$ forms a linearly dependent set if at least one of the vectors is a zero vector.*

Sol. Let $x_1 = \mathbf{0}$, then show that the vectors $x_1, x_2, \ldots\ldots, x_n$ are L.I.

Also let $\quad a \neq 0$ and $a_i = 0$ for $i = 2, 3, \ldots\ldots, n$

Then $\sum_{r=1}^{n} a_r x_r = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n = a_1 0 + 0x_2 + \ldots\ldots + 0x_n = 0$

$\Rightarrow \quad \sum_{r=1}^{n} a_r x_r = 0$ while $a_1 \neq 0$.

Hence the set of vectors $\{x_1, x_2, \ldots\ldots, x_n\}$ is L.D.

Example 23. *Consider the vector space $V(\mathbf{F})$ of polynomials in x and show that the infinite set $S = \{1, x, x^2, \ldots\ldots\}$ is L.I.*

Sol. Let $S_m = \{x^{n_1}, x^{n_2}, \ldots\ldots, x^{n_m}\}$ be any finite subset of S.

Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_m$ be scalars such that

$$\alpha_1 x^{n_1} + \alpha_2 x^{n_2} + \ldots\ldots + \alpha_m x^{n_m} = 0 \text{ (zero polynomial)}$$

$\Rightarrow \quad \alpha_1 = 0, \ \alpha_2 = 0, \ldots\ldots, \alpha_m = 0$

$\Rightarrow$ the subset S_m of S is L.I.

Hence S is L.I.

Example 24. *Examine whether $(1, -3, 5)$ belongs to the linear space generated by S, where $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$ or not?* (G.N.D.U. 1998, 85)

Sol. Suppose that $(1, -3, 5)$ belongs to the linear space generated by S.

$\therefore$ There exists scalars α, β, γ s.t.

$$(1, -3, 5) = \alpha(1, 2, 1) + \beta(1, 1, -1) + \gamma(4, 5, -2) \quad \ldots(1)$$

$$= (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma)$$

Comparing, $\quad \alpha + \beta + 4\gamma = 1 \quad \ldots(2)$

$\quad 2\alpha + \beta + 5\gamma = -3 \quad \ldots(3)$

and $\quad \alpha - \beta - 2\gamma = 5 \quad \ldots(4)$

Adding (2) and (4),

$$2\alpha + 2\gamma = 6 \quad \Rightarrow \quad \alpha + \gamma = 3 \quad \ldots(5)$$

Adding (3) and (4),

$$3\alpha + 3\gamma = 2 \quad \Rightarrow \quad \alpha + \gamma = \frac{2}{3} \quad \ldots(6)$$

(5) and (6) contradict each other.

Thus (1) is not possible.

Hence $(1, -3, 5)$ does not belong to the linear space of S.

Example 25. *Find the linearly independent subset A of the set $S = \{x_1, x_2, x_3, x_4\}$ in $V_3(\mathbf{R})$, where $x_1 = (1, 2, -1)$, $x_2 = (-3, -6, 3)$, $x_3 = (2, 1, 3)$, $x_4 = (8, 7, 7)$, which spans the same space as S.*

Sol. Since $x_1 \neq 0$ and $x_2 = -3x_1$,

$\therefore$ we exclude x_2 because it is a linear combination of x_1.

Again since x_3 cannot be expressed as a scalar multiple of x_1,

$\therefore$ we cannot exclude x_3.

Clearly $x_4 = 2x_1 + 3x_2$

$\Rightarrow$ x_4 is a linear combination of x_1 and x_3, therfore we exclude x_4.

Hence A = $\{x_1, x_3\}$ is the subset of S which is L.I.

10. Basis Set and Finite Dimensional

(*i*) **Def.** *Let V(F) be a vector space and S be a subset of V so that*

(*I*) *S consists of linearly independent vectors of V i.e., S is L.I.*

(*II*) *each vector of V is a linear combination of elements of S i.e., L(S) = V.*

Then S is called **basis set** *of V.*

This is denoted by **B.**

Remember. Zero vector can't be an element of a basis set.

Reason. A set of vectors having zero vector cannot be the basis of a vector space because such a set is L.D.

(*ii*) **Def.** *A vector space is said to be finite dimensional (or finitely generated) if there exists a finite subset S of V such that*

$$V = L(S).$$

THEOREMS

Theorem I. *In a finite dimensional vector space V(F) whose basis set is B = $\{x_1, x_2, \ldots\ldots, x_n\}$, every vector $x \in V$ is uniquely expressible as linear combination of the vectors in B.* (P.U. 1996)

Proof. Since B is a basis set of V, [*Given*]

$\therefore$ any vector $x \in$ V can be expressed as a linear combination of vectors in B

i.e., $x = \sum a_i x_i$ for α_i' s $\in$ F ...(1)

If possible, let $x = \sum \beta_i x_i$ for β_i' s $\in$ F ...(2)

be another representation.

Subtracting (2) from (1), we get :

$$0 = \sum \alpha_i x_i - \sum \beta_i x_i$$

$\Rightarrow$ $\sum(\alpha_i - \beta_i) x_i = 0$

$\Rightarrow$ $\alpha_i - \beta_i = 0$ for $i = 1, 2, \ldots, n$ [$\because$ *B, being basis set of V, is L.I.*]

$\Rightarrow$ $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, n$.

Thus the expressions (1) and (2) are same.

Hence each vector $x \in$ V can be uniquely expressible as linear combination of the vectors in B.

Note. Every vector x in finite dimensional vector space V(F) with a basis set B = $\{x_1, x_2, \ldots\ldots, x_n\}$ is expressed as

$$x = \sum a_i x_i$$

by unique set of scalars $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$.

These scalars can be taken as co-ordinates of the vectors x corresponding to the basis set as co-ordinate system.

Theorem II. (Existence Theorem)

There exists a basis for each finite dimensional vector space.

Proof. Let V (F) be a finite dimensional vector space.

Since V is finite dimensional, so it is a linear span of set S having finite number of vectors belonging to V.

Let $\quad S = \{x_1, x_2, ..., x_n\} \subset V$

and $\quad V = L(S)$ ...(1)

Without any loss of generality, we may assume that all vectors in S are non-zero, *i.e.*, $\mathbf{0} \notin S$, because contribution of $\mathbf{0}$ in the linear combination of elements of S is zero. Now as $S \subset V$, so either S is L.I. or L.D.

If S is L.I., then S will be a basis of V, which is the result.

If S is L.D., then there exists a vector, say x_k, in S which is linear combination of its preceding vectors in S, *i.e.*,

$$x_k = \sum_{i=1}^{k-1} \alpha_i x_i \quad \text{for } \alpha_i\text{'s} \in F \qquad ...(2)$$

Consider the set

$$S_1 = \{x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n\} \qquad ...(A)$$

Evidently $S_1 \subset S \quad \Rightarrow \quad L(S_1) \subset L(S)$

$\Rightarrow \quad L(S_1) \subset V$ ...(3)

[$\because$ *L(S) = V by (1)*]

Also if $x \in V \Rightarrow x = \sum_{j=1}^{n} \beta_j x_j$ for β_j's $\in F$

$$\Rightarrow x = \sum_{j \neq k} \beta_j x_j + \beta_k x_k$$

$$\Rightarrow x = \sum_{j \neq 1} \beta_j x_j + \beta_k \sum_{i=1}^{k-1} \alpha_i x_i \qquad [By\ (2)]$$

$$\Rightarrow x = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_{k-1} x_{k-1} + \beta_{k+1} x_{k+1} + ... + \beta_n x_n + \beta_k (\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{k-1} x_{k-1})$$

$$= (\beta_1 + \beta_k \alpha_1) x_1 + (\beta_2 + \beta_k \alpha_2) x_2 + ... + (\beta_{k-1} + \beta_k \alpha_{k-1}) x_{k-1} + [\beta_{k+1} x_{k+1} + ... + \beta_n x_n]$$

$$= \sum_{i=1}^{k-1} (\beta_i + \beta_k \alpha_i) x_i + \sum_{j=k+1}^{n} \beta_j x_j \quad \text{for each } (\beta_i + \beta_k \alpha_i)\text{'s}, \beta_j\text{'s} \in F$$

$\Rightarrow x$ is linear combination of elements of S_1

$\Rightarrow x \in L(S_1)$

$\Rightarrow V \subset L(S_1)$ ...(4)

But (3) and (4) $\Rightarrow V = L(S_1)$. ...(5)

Now as $S_1 \subset V$, so either S_1 is L.I. or L.D.

If S_1 is L.I., then S_1 becomes basis set of V.

But *if S_1 is L.D.*, then there exists a vector ; say x_l $(l > k) \in V$ in S_1, which is linear combination of its preceding vectors in S_1, *i.e.*,

$$x_l = \sum_{i \neq l} \gamma_i x_i \quad \text{for } \gamma_i\text{'s} \in F \qquad ...(6)$$

Now we consider the set

$$S_2 = \{x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_{l-1}, x_{l+1}, ..., x_n\}$$

and as proved above

$$V = L(S_2).$$

If S_1 *is L.I.*, then it is basis set of V. But if S_1 *is L.D.*, we repeat the procedure till we get a set which is L.I. and linear spanning set of V, thus giving basis of V. At the most by repeating the procedure we can get a basis set of V which contains only single non-zero vector, since the set of single non-zero vector is L.I.

Hence there exists a basis for each finite dimensional vector.

Remark. The above theorem clearly states that number of elements in the basis set of a finite dimensional vector space is less than or equal to number of elements in the linearly spanning set of V. Thus we can say that a finite dimensional vector space is a vector space with a finite basis set.

Theorem III. (Replacement Theorem)

If V (F) be a vector space which is spanned by a finite set S_1 *of vectors* $x_1, x_2, ..., x_n$, *then any linearly independent set of vectors in V contains not more than m elements.* (G.N.D.U. 1997, 96)

Proof. Let $S_1 = \{y_1, y_2, ..., y_n\}$

be L.I. set in V and as given

$$S_2 = \{x_1, x_2, ..., x_m\}$$

is a generating set of V *i.e.*, $L(S_1) = V$ ...(1)

To prove : $n \le m$, *i.e.*, any $(m+1)$ or more vectors in V are linearly dependent.

Since T is linearly spanned by S_1 and each element of S_2 is also an element of V, so each vector belonging to S_2 can be expressed as a linear combination of elements of S_1.

i.e., $$y_1 = \sum_{j=1}^{m} \alpha_j x_j \text{ for } \alpha_j\text{'s} \ne 0 \in F \quad ...(2)$$

Now consider the set $S_3 = \{y_1, x_1, x_2, ..., x_m\}$.

To prove : $L(S_3) \subset V$.

$$\forall x \in L(S_3) \Rightarrow x = \gamma_1 y_1 + \beta_1 x_1 + ... + \beta_m x_m = \gamma_1 \sum_{j=1}^{m} \alpha_j x_j + \beta_1 x_1 + ... + \beta_m x_m \quad [\textit{Using } (2)]$$

$$= (\gamma_1 \alpha_1 + \beta_1) x_1 + (\gamma_1 \alpha_2 + \beta_2) x_2 + ... + (\gamma_1 \alpha_m + \beta_m) x_m$$

$$= \delta_1 x_1 + \delta_2 x_2 + ... + \delta_m x_m, \qquad \text{where } \delta_i = \gamma_1 \alpha_i + \beta_i \in F, i = 1, 2, ..., m$$

$\Rightarrow \quad x \in L(S_1) = V.$

Hence $L(S_3) \subset V$ ...(3)

Evidently $S_1 \subset S_3 \Rightarrow L(S_1) \subset L(S_3)$

Thus $V \subset L(S_3)$ ...(4)

[*By (1)*]

(3) and (4) $\Rightarrow L(S_3) = V$.

Since one of the elements y_1 of S_3 is expressed as a linear combination of other elements of S_1 by (2), so the set S_3 is L.D. Thus one of the vectors in S_3 can be expressed as a linear combination of the preceding vectors and that vector cannot be y_1 as S_2 is L.I. So it must be one of the x_i's for $i = 1, 2, ..., m$. Let it be $x_k \in S_1$,

i.e., $x_k = \delta_1 y_1 + \gamma_1 x_1 + ... + \gamma_{k-1} x_{k-1}$ for $\delta_1, \gamma_1, ..., \gamma_{k-1} \in F$.

Now consider the set

$$S_4 = \{y_1, x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m\}.$$

We have $V = L(S_4)$.

Thus we have obtained a new set S_4 on replacing $x_k \in S_1$ by $y_1 \in S_2$.

Again consider the set $S_5 = \{y_2, y_1, x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots x_m\}$ and repeating the above arguments, we get the new set as

$$S_6 = \{y_2, y_1, \ldots x_{j-1}, x_j, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m\}$$

on replacing $x_j \in S_4$ by $y_2 \in S_2$, where $V = L(S_6)$.

Similarly we repeat the above procedure and in each step we are able to add one of the y_i's and delete one of the x_i's in the set S_1 to get a new linearly spanning set of V.

If $n \leq m$, then ultimately we obtain a linearly spanning set of V in the form

$$\{y_1, y_2, \ldots, y_n, x_{i\,1}, \ldots, x_{i\,m-n}\}.$$

If $n > m$, then after m steps we obtain a set

$$\{y_1, y_2, \ldots, y_m\} \quad \ldots(7)$$

which shows that y_{m+1} can be expressed as a linear combination of vectors $y_1, y_2, \ldots, y_m$. This implies that y_{m+1} is expressed as a linear combination of its preceding vectors in S_2 which contradicts the hypothesis that S_2 is L.I., thus showing that $n \not> m$.

Hence it is established that $n \leq m$, *i.e.*, number of elements in any L.I. set of V is less than or equal to the number of elements in a linearly spanning set of V.

Cor. 1. *If the set $S_1 = \{x_1, x_2, \ldots, x_m\}$ is a basis of vector space $V(F)$ and set $S_2 = \{y_1, y_2, \ldots, y_n\}$ in V is linearly independent, then $n \leq m$.*

Since S_1 is basis of V, so it is also linearly spanning set of V *i.e.*, $V = L(S_1)$. Hence by the theorem we have $n \leq m$.

Cor. 2. *If S_1 is a basis set of V, then S_4 is also basis set of V.*

Since $V = L(S_4)$ so to show that S_4 is a basis of V, we have only to show that S_4 is L.I. Assuming that S_4 is L.D., then there exists scalars $\beta_1, \beta_2, \ldots, \beta_n \in F$, such that

$$\sum_{j \neq i} \beta_j x_j + \beta_i y_1 = 0$$

with at least one $\beta_i \neq 0$. Evidently $\beta_i \neq 0$, because if $\beta_i = 0$, then $\sum_{j \neq i} \beta_j x_j = 0$, showing that each $\beta_j = 0$ because any subset of L.I. set S_1 is also L.I.

Thus $\quad y_1 = -\sum_{j \neq i} \beta_j \beta_i^{-1} x_i$. $\quad$ [*By inverse $\beta_i \neq 0$*]

Also $\quad y_1 = \sum_{j \neq i} \alpha_j x_j + \alpha_i x_i$ $\quad$ [*By (2)*]

Subtracting the two, we get

$$0 = \sum_{j \neq i} \left(\alpha_j + \frac{\beta_j}{\beta_i}\right) x_j + \alpha_i x_j$$

but as $\alpha_i \neq 0$, so it shows that S_1 is L.D. which is contradictory.

Hence S_4 is L.I.

Theorem IV. (Invariance of the number of elements in a basis).

If V (F) is a finite dimensional vector space, then any two bases of V have the same number of elements.
(G.N.D.U. 1996, 92 S, 87 S)

Proof. Let $B_1 = \{x_1, x_2, ..., x_m\}$
be one basis of V, *i.e.*, B_1 is L.I. and $V = L(B_1)$
and $B_2 = \{y_1, y_2, ..., y_n\}$ be another basis of V, *i.e.*, B_2 is L.I. and $V = L(B_2)$.

Consider that V is linearly spanned by B_1 (*i.e.*, $V = L(B_1)$ and B_2 as L.I. set in V, then we have

$$n \leq m \quad ...(1) \quad [\textit{Theorem III}]$$

Again considering $V = L(B_2)$ and B_1 as L.I. set in V, we have

$$m \leq n \quad ...(2) \quad [\textit{Theorem III}]$$

(1) and (2)
$\Rightarrow \quad m = n.$

Hence both the bases B_1 and B_2 of V have the same number of elements.

(*iii*) **Dimension of a vector space. Def.** *The number of elements in any basis set of a finite dimensional vector space V (F) is called the dimension of the vector space and is denoted as dim. V.*

Theorem V. *Each set consisting of (n +1) or more vectors in n-dimensional vector space V (F) is linearly dependent.*

Proof. Since V (F) is a vector space of dimension n,
$\therefore$ each basis set of V will contain exactly n vectors.

Now let $S_1 = \{y_1, y_2, ..., y_{n+1}\}$ be a L.I. subset of V (F). Then either this set itself is a basis set or it can be extended as a basis set. In both the cases the basis set will contain $(n + 1)$ or more vectors which contradicts the assumption that V is n-dimensional.

Hence S_1 is L.D. *i.e.*, any set containing $n + 1$ vectors is L.D.

Similarly we can say that any set containing more than $(n + 1)$ vectors belonging to V (F) will be L.D.

Cor. *In an n-dimensional vector space V (F) any L.I. set of n-vectors is a basis set of V (F).*

Since basis set of n-dimensional space consists of exactly n-independent vectors, therefore, any set of n-linearly independent vectors cannot be further extended to form a basis, but itself will act as a basis set of V (F).

Theorem VI. (Dimension of a sub-space)

If V (F) is a finite dimensional vector space of dimension n and if W is any sub-space of V, then W is a finite dimensional space having the dimension at the most n i.e.,

$$\dim. W \leq \dim. V. \qquad \text{(G.N.D.U. 1995 S, 92 S)}$$

Proof. Since dim. $V = n$, so any $(n + 1)$ or more vectors in V are L.D. Also W is vector sub-space of V, therefore, $W \subset V$ *i.e.*, every element of W is also an element of V, so any $(n + 1)$ or more vectors in W are L.D. Thus a L.I. set of vectors in W can obtain at most n elements.

Now we can find a largest set of L.I. vectors in W and let it be

$$S = \{x_1, x_2, ..., x_m\} \text{ for } m \leq n.$$

To prove : *S is a basis of W.* For this prove

(*i*) S is L.I. set in W.

(*ii*) For any $x \in W$ the set $S_1 = \{x, x_1, x_2, ..., x_m\}$ is L.D. (as S is the largest set of L.I. vectors in W).

Thus $\alpha x + \alpha_1 x_1 + ...+ \alpha_m x_m = 0$ for α, α_i's $0 \in F$ ...(1)

where all α_i's are not equal to zero.

Thus $\alpha \neq 0$ because if $\alpha = 0$, then

$$\sum_{i=1}^{m} \alpha_i x_i = 0 \Rightarrow \text{ each } \alpha_i = 0, \qquad [\because S \text{ is } L.I.]$$

which is a contradiction to the assumption.

Thus as $\alpha \neq 0$ so by (1), we have

$$x = -\alpha^{-1}\alpha_1 x_1 - \alpha^{-1}\alpha_2 x_2 - \ldots - \alpha^{-1}\alpha_m x_m$$
$$= (-\alpha^{-1}\alpha_1) x_1 + (-\alpha^{-1}\alpha_2) x_2 + \ldots + (-\alpha^{-1}\alpha_m) x_m,$$

which shows that each vector in W can be expressed as a linear combination of elements of S *i.e.*, W = L (S). By this we can say that W is a finite dimensional because its linearly spanning set S contains finite number of elements.

Hence by (*i*) and (*ii*) S is the basis set of W, so dim. W = m, where $m \leq n$,

i.e., dim. W $\leq$ dim. V.

Theorem VII. (Dimension of a linear sum)

If W_1 and W_2 are subspaces of a finite dimensional vector space V (F), then

dim. $(W_1 + W_2) =$ *dim.* W_1 + *dim.* W_2 – *dim.* $(W_1 \cap W_2)$.

(G.N.D.U. 1998, 92 ; P.U. 1998 ; Pbi. U. 1996, 87)

Proof. Since V is a finite dimensional vector space $W_1 + W_2$, $W_1 \cap W_2$ are vector subspaces of V so $W_1 + W_2$, $W_1 \cap W_2$ are finite dimensional.

Let $B_1 = \{z_1, z_2, \ldots\ldots, z_r\}$

be the basis of $W_1 \cap W_2$, then dim $W_1 \cap W_2 = r$ ...(1)

Since $W_1 \cap W_2$ is a subspace of W_1 and W_2 so the basis set B_1 can be extended as a basis of W_1 and W_2.

Then sets $B_2 = \{z_1, z_2, \ldots\ldots, z_r, x_1, x_2, \ldots, x_{m-r}\}$

$B_3 = \{z_1, z_2, \ldots\ldots, z_r, y_1, y_2, \ldots, x_{n-r}\}$

are bases of W_1 and W_2 respectively.

Since B_2 and B_3 are bases of W_1 and W_2, [*Supposed*]

$\therefore$ dim $W_1 = m$ and dim. $W_2 = n$ ...(2)

Consider the set

$S = B_2 \cup B_3 = \{z_1, z_2, \ldots, z_r, x_1, x_2, \ldots, x_{m-r}, y_1, y_2, \ldots, y_{n-r}\}$

To prove : S is a basis of $W_1 + W_2$.

(*i*) Firstly *S is L.I.*

Suppose $\displaystyle\sum_{i=1}^{r} \alpha_i z_i + \sum_{j=1}^{m-r} \beta_j x_j + \sum_{k=1}^{n-r} \gamma_k y_k = 0$ ...(3),

where α_i's, β_j's, γ_k's $\in$ F

$\Rightarrow$ $\Sigma \beta_j x_j + \Sigma \alpha_i z_i = -\Sigma \gamma_k y_k$

$\Rightarrow$ linear combination of elements of B_2

$\Rightarrow$ $\Sigma \gamma_k y_k \in W_1$ (B_2 being basis of W_1).

(But also $\Sigma \gamma_k y_k = \Sigma \gamma_k y_k + \Sigma 0_i z_i$ for 0_i being additive identity 0 of F

= linear combination of elements of B_2)

$\Rightarrow$ $\Sigma \gamma_k y_k \in W_1 \cap W_2$ (As B_3 is basis of W_2, so $\Sigma \gamma_k y_k \in W_2$ also)

$\Rightarrow$ $\displaystyle\Sigma \gamma_k y_k = \sum_{i=1}^{r} \delta_i z_i$

(Since B_1 is a basis of $W_1 \cap W_2$, so each element of $W_1 \cap W_2$ can be expressed as linear combination of elements of B_1)

$\Rightarrow \quad \Sigma \gamma_k\, y_k - \Sigma \delta_i z_i = 0$

$\Rightarrow \quad \gamma_1 = 0, \gamma_2 = 0, \ldots, \gamma_{n-r} = 0, \delta_1 = 0, \delta_2 = 0, \ldots, \delta_r = 0$ ...(4)

(Since B_2 is L.I. being basis set of W_2)

As each $\gamma_k = 0$ so by (3), we have

$$\Sigma \alpha_i z_i + \Sigma \beta_j x_j = \Sigma 0\, y_k = 0$$

$\Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_r = 0, \beta_1 = 0, \beta_2 = 0, \ldots, \beta_{m-r} = 0$ ...(5)

(Since B_2 is L.I. being basis of W_1)

Hence by (4) and (5) each α_i, β_j and γ_k is zero, S is L.I.

(*ii*) Secondly $W_1 + W_2 = L(S)$.

Let $z \in L(S) \Rightarrow z = \Sigma \alpha_i z_i + \Sigma \beta_j x_j + \Sigma \gamma_k y_k$

$\Rightarrow z = (\Sigma \alpha_i z_i + \Sigma \beta_j x_j) + (\Sigma \delta_i z_i + \Sigma \gamma_k y_k)$

$\Rightarrow z = (a \text{ l. c. of } B_2) + (a \text{ l. c. of } B_3)$

$\Rightarrow z = x + y \text{ for } x \in W_1, y \in W_2$

$\Rightarrow z \in W_1 + W_2$.

Thus $L(S) \subset W_1 + W_2$ ...(6)

Also let $z \in W_1 + W_2 \quad \Rightarrow \quad z = x + y \text{ for } x \in W_1, y \in W_2$

$\Rightarrow \quad z = (\Sigma \alpha_i z_i + \Sigma \beta_j x_j) + (\Sigma \delta_i y_i + \Sigma \gamma_k y_k)$

(As B_2, B_3 are bases of W_1 and W_2)

$\Rightarrow \quad z = \Sigma (\alpha_i + \delta_i) z_i + \Sigma \beta_j x_j + \Sigma \gamma_k y_k$

$\Rightarrow \quad z = (a \text{ l. c. of elements of S}) \quad \Rightarrow \quad z \in L(S)$.

Thus $W_1 + W_2 \subset L(S)$ ...(7)

(6) and (7) $\Rightarrow$ $W_1 + W_2 = L(S)$.

Thus S is a basis set of $W_1 + W_2$, which shows that

dim $(W_1 + W_2) = r + m - r + n - r = m + n - r$ ...(8)

i.e., $\quad m + n - r = (m) + (n) - r$

Hence dim. $(W_1 + W_2) =$ dim. W_1 + dim. W_2 − dim $(W_1 \cap W_2)$. [*From (1), (2) and (8)*]

SOLVED EXAMPLES

Example 1. *Examine whether the following set of vectors in $V_3(R)$ forms a basis or not :*

(*i*) (*1, 0, 0*), (*0, 1, 0*), (*0, 0, 1*)

(*ii*) (*1, 1, 2*), (*1, 2, 5*), (*5, 3, 4*) (P.U. 1992)

(*iii*) (*1, 2, 1*), (*2, 1, 0*), (*1, − 1, 2*)

(*iv*) (*1, 0, − 1*), (*1, 2, 1*), (*0, − 3, 2*)

(*v*) (*1, 0, 0*), (*1, 1, 0*), (*1, 1, 1*), (*0, 1, 0*).

Sol. (*i*) Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

S will be a basis of V_3 **(R)** if S is L.I. and $L(S) = V_3$ **(R)**.

I. *S is L.I.*

Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$\alpha\,(1, 0, 0) + \beta\,(0, 1, 0) + \gamma\,(0, 0, 1) = \mathbf{0}$$

$\Rightarrow \quad (\alpha, 0, 0) + (0, \beta, 0) + (0, 0, \gamma) = (0, 0, 0)$

$\Rightarrow \quad (\alpha, \beta, \gamma) = (0, 0, 0) \quad \Rightarrow \quad \alpha = 0, \beta = 0, \gamma = 0$

Thus S is L.I.

II. $L\,(S) = V_2\,(\mathbf{R})$.

We know that $L\,(S) \subset V_3\,(\mathbf{R})$ ...(1)

Let (a, b, c) be any vector in $V_3\,(\mathbf{R})$, where $a, b, c \in \mathbf{R}$, then as proved above

$$(a, b, c) = a\,(1, 0, 0) + b\,(0, 1, 0) + c\,(0, 0, 1)$$

i.e., each element in V_3 can be expressed as a linear combination of elements of S

$\therefore \quad V_3\,(\mathbf{R}) \subset L\,(S)$ ...(2)

(1) and (2) $\Rightarrow L\,(S) = V_3\,(\mathbf{R})$.

Hence S is a basis set of $V_3\,(\mathbf{R})$.

(*ii*) Let $S = \{(1, 1, 2), (1, 2, 5), (5, 3, 4)\}$

S will be a basis of $V_3\,(\mathbf{R})$ if S is L.I. and $L\,(S) = V_3\,(\mathbf{R})$.

I. *S is L.I.*

Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that $\alpha\,(1, 1, 2) + \beta\,(1, 2, 5) + \gamma\,(5, 3, 4) = \mathbf{0}$

$\Rightarrow \quad (\alpha, \alpha, 2\,\alpha) + (\beta, 2\,\beta, 5\,\beta) + (5\,\gamma, 3\,\gamma, 4\,\gamma) = (0, 0, 0)$

$\Rightarrow \quad (\alpha + \beta + 5\,\gamma, \alpha + 2\,\beta + 3\,\gamma, 2\,\alpha + 5\,\beta + 4\,\gamma) = (0, 0, 0)$

$\Rightarrow \quad \alpha + \beta + 5\,\gamma = 0, \alpha + 2\,\beta + 3\,\gamma = 0, 2\,\alpha + 5\,\beta + 4\,\gamma = 0$ ...(A)

These equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\quad A\,X = O$, where $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$

Now $|\,A\,| = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix} = 1\,(8 - 15) - 1\,(4 - 6) + 5\,(5 - 4)$

$= -7 + 2 + 5 = 0$

$\therefore \quad$ Equation (A) have a non-trivial solution

i.e., α, β, γ are real and are not all zero.

$\Rightarrow \quad$ vectors (1, 1, 2), (1, 2, 5), (5, 3, 4) are L.D.

Hence the set S is not a basis of $V_3\,(\mathbf{R})$.

(*iii*) Let $S = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$

S will be a basis of $V_3\,(\mathbf{R})$ if S is L.I. and $L\,(S) = V_3\,(\mathbf{R})$

I. *S is L.I.*

Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$\alpha\,(1, 2, 1) + \beta\,(2, 1, 0) + \gamma\,(1, -1, 2) = \mathbf{0}$$

$\Rightarrow \quad (\alpha, 2\alpha, \alpha) + (2\beta, \beta, 0) + (\gamma, -\gamma, 2\gamma) = (0, 0, 0)$

$\Rightarrow \quad (\alpha + 2\beta + \gamma, 2\alpha + \beta - \gamma, \alpha + 2\gamma) = (0, 0, 0)$

$\Rightarrow \quad \alpha + 2\beta + \gamma = 0, 2\alpha + \beta - \gamma = 0, \ \alpha + 2\gamma = 0$ ...(A)

$\Rightarrow \quad \alpha = 0, \beta = 0, \gamma = 0.$

These equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\quad$ AX = O, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

$$\text{Now } |A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 1\,(2 + 0) - 2\,(4 + 1) + 1\,(0 - 1)$$

$$= 2 - 10 - 1 = -9 \neq 0$$

$\therefore$ Equation (A) have only a trivial solution.

$\therefore \quad \alpha = 0, \beta = 0, \gamma = 0.$

$\Rightarrow$ Thus S is L.I.

II. $L(S) = V_3$ **(R)**

We know that L (S) $\subset$ V_3 **(R)** ...(1)

Let (a, b, c) be any vector in V_3 **(R)**, where $a, b, c \in \mathbf{R}$, then

$(a, b, c) = a\,(1, 0, 0) + b\,(0, 1, 0) + c\,(0, 0, 1)$ ...(2),

where $\quad (1, 0, 0) = x\,(1, 2, 1) + y\,(2, 1, 0) + z\,(1, -1, 2)$

$= (x, 2x, x) + (2y, y, 0) + (z, -z, 2z) = (x + 2y + z, 2x + y - z, x + 2z)$

$\therefore \ x + 2y + z = 1, 2x + y - z = 0, x + 2z = 0$

Solving, we get $x = -\dfrac{2}{9}, y = \dfrac{5}{9}, z = \dfrac{1}{9}$.

$\therefore \quad (1, 0, 0) = -\dfrac{2}{9}\,(1,2,1) + \dfrac{5}{9}\,(2, 1, 0) + \dfrac{1}{9}(1, -1, 2)$

Similarly, $(0, 1, 0) = \dfrac{4}{9}(1, 2, 1) - \dfrac{1}{9}(2, 1, 0) - \dfrac{2}{9}(1, -1, 2)$

and $\quad (0, 0, 1) = \dfrac{1}{3}\,(1, 2, 1) - \dfrac{1}{3}\,(2,1,0) + \dfrac{1}{3}\,(1, -1, 2)$

Thus from (2) on putting, we get

$$(a, b, c) = a\left[-\frac{2}{9}\,(1, 2, 1) + \frac{5}{9}\,(2, 1, 0) + \frac{1}{9}\,(1, -1, 2)\right]$$

$$+ b\left[\frac{4}{9}(1, 2, 1) - \frac{1}{9}\,(2, 1, 0) - \frac{2}{9}(1, -1, 2)\right] + c\left[\frac{1}{3}\,(1, 2, 1) - \frac{1}{3}(2, 1, 0) + \frac{1}{3}(1, -1, 2)\right]$$

$$= \left(-\frac{2}{9}\,a + \frac{4}{9}\,b + \frac{1}{3}c\right)(1, 2, 1) + \left(\frac{5a}{9} - \frac{b}{9} - \frac{c}{3}\right)(2,1,0) + \left(\frac{a}{9} - \frac{2b}{9} + \frac{c}{3}\right)(1, -1, 2)$$

i.e., each element in V_3 can be expressed as a linear combination of elements of S

V_3 **(R)** $\subset$ L (S) ...(4)

(1) and (3) $\Rightarrow$ L (S) = V_3 **(R)**.

Hence S is a basis set of V_3 **(R)**.

(iv) Let $S = \{(1, 0, -1), (1, 2, 1), (0, -3, 2)\}$

Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$\alpha (1, 0, -1) + \beta (1, 2, 1) + \gamma (0, -3, 2) = \mathbf{0}$$

$\Rightarrow$ $(\alpha, 0, -\alpha) + (\beta, 2\beta, \beta) + (0, -3\gamma, 2\gamma) = (0, 0, 0)$

$\Rightarrow$ $(\alpha + \beta, 2\beta - 3\gamma, -\alpha + \beta + 2\gamma) = (0, 0, 0)$

$\Rightarrow$ $\alpha + \beta = 0, \quad 2\beta - 3\gamma = 0, \quad -\alpha + \beta + 2\gamma = 0$

Solving, $\alpha = 0, \beta = 0, \gamma = 0$

$\Rightarrow$ S is L.I.

Hence S forms a basis of V_3 (**R**).

(v) Let $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$

(I) *S is L.I.*

Let $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ such that

$$\alpha (1, 0, 0) + \beta (1, 1, 0) + \gamma (1, 1, 1) + \delta (0, 1, 0) = \mathbf{0}$$

$\Rightarrow$ $(\alpha, 0, 0) + (\beta, \beta, 0) + (\gamma, \gamma, \gamma) + (0, \delta, 0) = (0, 0, 0)$

$\Rightarrow$ $(\alpha + \beta + \gamma, \beta + \gamma + \delta, \gamma) = (0, 0, 0)$

$\Rightarrow$ $\alpha + \beta + \gamma = 0, \ \beta + \gamma + \delta = 0, \gamma = 0$

Here we have three equations for finding out four unknowns.

So it will have non-zero solution.

Thus S is not L.I.

Hence S is not a basis set of V_3 (**R**).

Example 2. *Prove that the set of vectors*

$$(1, 0, 0, \ldots\ldots, 0), (0, 1, 0, \ldots\ldots, 0), \ldots\ldots, (0, 0, 0, \ldots\ldots, 1)$$

in V_n (**R**) *is a basis set.*

Sol. Let $S = \{(1, 0, 0, \ldots\ldots, 0), (0, 1, 0, \ldots\ldots, 0), \ldots\ldots, (0, 0, 0, \ldots\ldots, 1)\}$

(I) *S is L.I.*

Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in \mathbf{R}$ such that

$$\alpha_1 (1, 0, 0, \ldots\ldots, 0) + \alpha_2 (0, 1, 0, \ldots\ldots, 0) + \ldots\ldots + \alpha_n (0, 0, 0, \ldots\ldots, 1) = \mathbf{0}$$

$\Rightarrow$ $(\alpha_1, 0, 0, \ldots\ldots, 0) + (0, \alpha_2, 0, \ldots\ldots, 0) + \ldots\ldots + (0, 0, 0, \ldots\ldots, \alpha_n) = (0, 0, 0, \ldots\ldots, 0)$

$\Rightarrow$ $(\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n) = (0, 0, 0, \ldots\ldots, 0)$

$\Rightarrow$ $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \ldots\ldots, \alpha_n = 0.$

Thus S is L.I.

(II) $L(S) = V_n$ (**R**).

Let $(a_1, a_2, \ldots\ldots, a_n)$ be any vector is V_n (**R**).

We can write $(a_1, a_2, a_3, \ldots\ldots, a_n) = a_1 (1, 0, 0, \ldots, 0) + a_2 (0, 1, 0, \ldots, 0) + \ldots + a_n (0, 0, 0, \ldots, 1)$.

This shows that $V_n = L(S)$.

Hence S is a basis set of V_n (**R**).

Example 3. *Show that the set*

$$S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$$

is a basis of vector space V_3 (**C**).

Hence find the co-ordinates of the vector $(3 + 4i, 6i, 3 + 7i)$ *in* V_3 (**C**) *with respect to S.*

Sol. As in Ex. 1, S is a basis set of V_3 **(C)**.

Now let $(3+4i, 6i, 3+7i) = \alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)$, where $\alpha,\beta,\gamma \in C$

$$= (\alpha,0,0)+(\beta,\beta,0)+(\gamma,\gamma,\gamma) = (\alpha+\beta+\gamma, \beta+\gamma, \gamma)$$

Then $3+4i = \alpha+\beta+\gamma$ ...(1)

$6i = \beta+\gamma$ ...(2)

$3+7i = \gamma$ ...(3)

From (3), $\gamma = 3+7i$

Putting in (2), $6i = \beta+3+7i \Rightarrow \beta = -i-3$

Putting in (1), $3+4i = \alpha - i - 3 + 3 + 7i \Rightarrow \alpha = 3-2i$

Hence the required co-ordinates are $(3-2i, -3-i, 3+7i)$.

Example 4. *Let* $S = \{(1, 2, -1), (2, -3, 2)\}$ *and* $T = \{(4, 1, 3), (-3, 1, 2)\}$, *show that* $L(S) \neq L(T)$. (P.U. 1992)

Sol. If possible, let L (S) = L (T).

Consider any $v \neq 0 \in L(S)$, then

$v = \alpha_1 v_1 + \alpha_2 v_2$ for some $v_1, v_2 \in R$, not both zero.

Now L (S) = L (T) $\Rightarrow v \in L(T)$

$\Rightarrow v = \alpha_3 v_3 + \alpha_4 v_4$ for some $\alpha_3, \alpha_4 \in R$, not both zero.

$\therefore \quad \alpha_1 v_1 + \alpha_2 v_2 = \alpha_3 v_3 + \alpha_4 v_4$

$\Rightarrow \alpha_1(1,2,-1)+\alpha_2(2,-3,2) = \alpha_3(4,1,3)+\alpha_4(-3,1,2)$

$\Rightarrow \quad \alpha_1 + 2\alpha_2 = 4\alpha_3 - 3\alpha_4$ ...(1)

$2\alpha_1 - 3\alpha_2 = \alpha_3 + \alpha_4$ ...(2)

and $-\alpha_1 + 2\alpha_2 = 3\alpha_3 + 2\alpha_4$ ...(3)

Solving (1) and (3), we get

$$\alpha_1 = \frac{1}{2}(\alpha_3 - 5\alpha_4) \text{ and } \alpha_2 = \frac{1}{4}(7\alpha_3 - \alpha_4)$$

Putting in (2), we get $2\alpha_1 - 3\alpha_2 = \alpha_3 - 5\alpha_4 - \frac{3}{4}(7\alpha_3 - \alpha_4)$

$$= -\frac{17}{4}(\alpha_3 + \alpha_4) \neq \alpha_3 + \alpha_4.$$

Thus (2) is not satisfied.

Hence L (S) $\neq$ L(T).

Example 5. (*a*) *Show that (1, 0, 0), (1, 1, 0), (1, 1, 1) in* R^3 *(R) is a basis. Hence find the co-ordinate vector of (a, b, c)* $\in R^3$ *relative to this basis.* (G.N.D.U. 1990)

(*b*) *Show that the set of vectors (0, 1, – 1), (1, 1, 0), (1, 0, 2) is a basis of* R^3 *(R). Hence find the co-ordinate vector of (1, 0, – 1) w.r.t. this basis.* (Pbi. U. 1996 ; G.N.D.U. 1990)

Sol. (*a*) As in Ex. 1 (*i*), (1, 0, 0), (1, 1, 0), (1, 1, 1) in R^3 (R) is the basis.

Now let $(a,b,c) = \alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)$,

where $\alpha,\beta,\gamma \in R = (\alpha,0,0)+(\beta,\beta,0)+(\gamma,\gamma,\gamma) = (\alpha+\beta+\gamma, \beta+\gamma, \gamma)$

Then $a = \alpha+\beta+\gamma$ ...(1)

$b = \beta+\gamma$ ...(2)

$c = \gamma$ ...(3)

From (3), $\gamma = c$.

Putting in (2), $b = \beta + c \Rightarrow \beta = b - c$.

Putting in (1), $a = \alpha + b \Rightarrow \alpha = a - b$.

Hence the required co-ordinates are $(a - b, b - c, c)$.

(*b*) Let $S = \{(0, 1, -1), (1, 1, 0), (1, 0, 2)\}$.

S will be basis of R^3 (R) if S is L.I. and $L(S) \equiv L(R^3)$.

(I) *S is L.I.*

Let $\alpha, \beta, \gamma \in R$ such that $\alpha(0, 1, -1) + \beta(1, 1, 0) + \gamma(1, 0, 2) = \mathbf{0}$

$\Rightarrow \quad (0, \alpha, -\alpha) + (\beta, \beta, 0) + (\gamma, 0, 2\gamma) = (0, 0, 0)$

$\Rightarrow \quad (\beta + \gamma, \alpha + \beta, -\alpha + 2\gamma) = (0, 0, 0)$

$\Rightarrow \quad \beta + \gamma = 0, \alpha + \beta = 0, -\alpha + 2\gamma = 0$

$\Rightarrow \quad \alpha = 0, \beta = 0, \gamma = 0.$

Thus S is L.I.

II. $L(S) = R^3(R)$.

We know that $L(S) \subset R^3(R)$ ...(1)

Let (a, b, c) be any vector in R^3 (R), where $a, b, c \in R$, then as proved above

$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$ ...(2),

where $(1, 0, 0) = x(0, 1, -1) + y(1, 1, 0) + z(1, 0, 2) = (0, x, -x) + (y, y, 0) + (z, 0, 2z)$

$= (y + z, x + y, -x + 2z)$.

$\therefore \quad y + z = 1, x + y = 0, -x + 2z = 0.$

Solving, we get

$x = -2, y = 2, z = -1.$

$\therefore \quad (1, 0, 0) = -2(0, 1, -1) + 2(1, 1, 0) - 1.(1, 0, 2)$

Similarly $(0, 1, 0) = 2(0, 1, -1) - 1.(1, 1, 0) + 1.(1, 0, 2)$

and $(0, 0, 1) = 1.(0, 1, -1) - 1.(1, 1, 0) + 1.(1, 0, 2)$

Thus from (2), on putting, we get

$$(a, b, c) = a[-2(0, 1, -1) + 2(1, 1, 0) - 1.(1, 0, 2)]$$
$$+ b[2(0, 1, -1) - 1.(1, 1, 0) + 1.(1, 0, 2)] + c[1.(0, 1, -1) - 1.(1, 1, 0) + 1.(1, 0, 2)]$$
$$= (-2a + 2b + c)(0, 1, -1) + (2a - b - c)(1, 1, 0) + (-a + b + c)(1, 0, 2)$$

i.e., each element in R^3 can be expressed as a linear co-ordinate of S.

$\therefore \quad R^3(R) \subset L(S)$ ...(3)

(1) and (3) $\Rightarrow L(S) = R^3(R)$

Hence S is the basis set of R^3 (R).

Now let $(1, 0, -1) = \alpha(0, 1, -1) + \beta(1, 1, 0) + \gamma(1, 0, 2)$, where $\alpha, \beta, \gamma \in R$

$= (0, \alpha, -\alpha) + (\beta, \beta, 0) + (\gamma, 0, 2\gamma)$

$= (\beta + \gamma, \alpha + \beta, -\alpha + 2\gamma)$

Then $1 = \beta + \gamma$ ...(4)

$0 = \alpha + \beta$ ...(5)

$-1 = -\alpha + 2\gamma$ ...(6)

(5) – (4) gives $-1 = \alpha - \gamma$ ...(7)

(6) + (7) gives $-2 = \gamma \Rightarrow \gamma = -2.$

Putting in (4), $1 = \beta - 2 \Rightarrow \beta = 3$.

Putting in (5), $0 = \alpha + 3 \Rightarrow \alpha = -3$.

Hence the required co-ordinates are $(-3, 3, -2)$.

Example 6. *If V be the vector space of ordered pairs of complex numbers over the real field* **R**, *then show that the set*

$$S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$$

is basis of V **(R)**.

Sol. We have $S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$.

(I) *S is L.I.*

Let $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ such that

$$\alpha (1, 0) + \beta (i, 0) + \gamma (0, 1) + \delta (0, i) = \mathbf{0}$$

$\Rightarrow \quad (\alpha, 0) + (i\beta, 0) + (0, \gamma) + (0, i\delta) = (0, 0)$

$\Rightarrow \quad (\alpha + i\beta, \gamma + i\delta) = (0, 0)$

$\therefore \quad \alpha + i\beta = 0 \quad \ldots (1) \quad$ and $\quad \gamma + i\delta = 0 \quad \ldots(2)$

[$\because$ *When $a + ib = c + id$, for $a, b, c, d \in \mathbf{R}$, then $a = c$, $b = d$*]

(1) $\Rightarrow \alpha = 0, \beta = 0$ and (2) $\Rightarrow \gamma = 0, \delta = 0$

$\therefore \quad \alpha = 0, \beta = 0, \quad \gamma = 0$ and $\delta = 0$.

Thus S is L.I.

(II) $L(S) = V$ **(R)**.

Let x be any vector in V **(R)**, then

$$x = (a + ib, c + id) \text{ for } a, b, c, d \in \mathbf{R}$$

Now $\quad x = (a + ib, c + id) = a(1, 0) + b(i, 0) + c(0, 1) + d(0, i)$

$\Rightarrow \quad x$ is a linear combination of elements of S.

Thus L (S) = V **(R)**.

Hence S is a basis set of V (R).

Example 7. *Show that the set $S = \{1, x, x^2, \ldots\ldots, x^n\}$ of $(n + 1)$ polynomials is a basis set for the vector space P_n* **(R)** *of all polynomials of degree at the most n over the field of real numbers.*

Sol. We have $S = \{1, x, x^2, \ldots\ldots, x^n\}$

(I) *S is L.I.*

Let $a_0, a_1, a_2, \ldots\ldots, a_n \in \mathbf{R}$ such that

$$a_0 \cdot 1 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = 0 \qquad \text{(zero polynomial)}$$

$\Rightarrow \quad a_0 \cdot 1 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \ldots\ldots 0 \cdot x^n$

$\Rightarrow \quad a_0 = 0, a_1 = 0, a_2 = 0, \ldots\ldots, a_n = 0.$

Thus S is L.I.

(II) Since any polynomial in P_n **(R)**, say

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$$

is a linear combination of the element of S.

Hence S is a basis set of P_n **(R)**.

Example 8. *Extend $\{(3, -1, 2)\}$ to two different bases for R^3.* (G.N.D.U. 1985)

Sol. It is known that the vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$ form a standard basis of R^3 **(R)**.

Further, since the vector v $(3, -1, 2)$ of R^3 **(R)** is non-zero L.I.,

$\therefore$ v, v_1, v_2, v_3 span R^3.

And any basis of R^3 contains exactly 3 linearly independent elements

[$\because$ *R^3 is 3-dimensional vector space*]

To check : v, v_1, v_2 are L.I. or not.

We form the matrix A having vectors v, v_1, v_2 and then reduce it to echelon form*. [* *Art. 11*]

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad [\textit{Operating } 3R_2]$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} \qquad [\textit{Operating } R_2 \to R_{21}(-1)]$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad [\textit{Operating } R_3 \to R_{32}(-1)]$$

Thus the echelon form of A has no non-zero rows

$\Rightarrow$ v, v_1, v_2 are L.I. over R.

Hence v, v_1, v_2 form the basis of R^3.

$\Rightarrow$ the vectors $(3, -1, 2), (0, 1, -2)$ and $(0, 0, 2)$ are L.I. over R

$\Rightarrow$ these vectors form a basis of R^3.

Hence $\{(3, -1, 2)\}$ has been extended to two bases :

(*i*) $\{(3, -1, 2), (1, 0, 0), (0, 1, 0)\}$ and

(*ii*) $\{(3, -1, 2), (0, 1, -2), (0, 0, 2)\}$.

Example 9. *Find the basis and dimension of the sub-space W of R^4, generated by $(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)$.* (P.U. 1998 ; G.N.D.U. 1993, 85)

Also extend the basis of W to a basis of the whole space R^4.

Sol. Let $A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-1) \textit{ and } R_{31}(-3)]$$

$$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_{32}(-4)]$$

which is of echelon form.

The non-zero rows of this matrix form the base of the row space of A.

Thus the vectors $(1, -4, -2, 1)$ and $(0, 1, 1, 1)$ form a basis of W and dim. W = 2.

Since R^4 is 4-dimensional,

$\therefore$ we require two more L.I. vectors in addition to the above two vectors.

The vectors $(1, -4, -2, 1)$, $(0, 1, 1, 1)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are L.I. over R.

[$\because$ *they form echelon matrix*]

and consequently these form a basis of R^4.

Also this is an extension of the basis of W.

Example 10. (*a*) *Prove that the polynomials* $1, 2-x, 3+x^2, 4-x^3$ *span the subspace* W *of all polynomials over reals and of degree* ≤ 3. (P.U. 1985)

(*b*) *Prove that the polynomials* $1, 1+t, 1+t^2, 1+t^3$ *span the subspace* W *of polynomials in* t *of degree* ≤ 3 *(including zero polynomial) over* **R**.

Sol. (*a*) Since W is a subspace of all polynomials in x over **R** of degree ≤ 3 (including zero polynomial),

$\therefore$ dim. W = 4

$\Rightarrow$ any four L.I. polynomials of W form its basis.

Thus the given polynomials $1, 2-x, 3+x^2, 4-x^3$ span all of W iff they are L.I. over **R**.

To prove L.I. : Let $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ such that

$$\alpha.1 + \beta(2-x) + \gamma(3+x^2) + \delta(4-x^3) = 0$$

$\Rightarrow$ $(\alpha + 2\beta + 3\gamma + 4\delta) - \beta x + \gamma x^2 - \delta x^3 = 0$

$\Rightarrow$ $\alpha + 2\beta + 3\gamma + 4\delta = 0, -\beta = 0, \gamma = 0, -\delta = 0$

$\Rightarrow$ $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0$

$\Rightarrow$ Given polynomials are L.I. over **R**.

(*b*) Proceed as in part (*a*).

Example 11. *Let* $V(\mathbf{R}) = \mathbf{P}_n$ *be the vector space of polynomials in* t*, over the field of reals, of degree* $\leq n$. *Determine whether or not each of the following is a basis of* V *:*

(*i*) $\{1, t, t^2, \ldots\ldots, t^n\}$

(*ii*) $\{1+t, t+t^2, t^2+t^3, \ldots\ldots, t^{n-2}+t^{n-1}, t^{n-1}+t^n\}$

(*iii*) $\{1, 1+t, 1+t+t^2, 1+t+t^2+t^3, \ldots\ldots, 1+t+t^2+\ldots\ldots+t^{n-1}+t^n\}$.

Sol. We have

$$\mathbf{P}_n = V(\mathbf{R}) = \{p(t) : p(t) = a_0 + a_1 t + \ldots\ldots + a_n t^n \text{ for } a_0, a_1, \ldots\ldots, a_n \in \mathbf{R}\}$$

(*i*) Here $S = \{1, t, t^2, \ldots\ldots, t^n\}$.

(I) *S is L.I.*

Let $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in \mathbf{R}$ such that

$$\alpha_0 . 1 + \alpha_1 . t + \ldots + \alpha_n . t^n = 0 \qquad \text{(zero polynomial)}$$

$\Rightarrow$ $\alpha_0 . 1 + \alpha_1 . t + \ldots\ldots + \alpha_n t^n = 0. 1 + 0. t + \ldots + 0. t^n$

$\Rightarrow$ $\alpha_0 = 0, \alpha_1 = 0, \ldots\ldots, \alpha_n = 0$.

Thus S is L.I.

(II) $L(S) = V(\mathbf{R})$.

Let $a_0 + a_1 t + \ldots\ldots + a_n t^n$ be any polynomial in $V_n(\mathbf{R})$.

Then $a_0 + a_1 t + \ldots\ldots + a_n t^n = a_0 . 1 + a_1 t + \ldots\ldots + a_n t^n$

$\Rightarrow$ it is a linear combination of elements of S

Thus L (S) = V (**R**).

Hence S is a basis set of V (**R**) = $\mathbf{P}_n$.

(*ii*) Here $S = \{1+t, t+t^2, t^2+t^3, \ldots\ldots, t^{n-2}+t^{n-1}, t^{n-1}+t^n\}$.

(I) *S is L.I.*

Let $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots\ldots\ldots, \alpha_n \in \mathbf{R}$ such that

$\alpha_0(1+t)+\alpha_1(t+t^2)+\alpha_2(t^2+t^3)+\ldots\ldots+\alpha_{n-1}(t^{n-2}+t^{n-1})+\alpha_n(t^{n-1}+t^n)=0$ (zero polynomial)

$\Rightarrow \quad \alpha_0+(\alpha_0+\alpha_1)t+(\alpha_1+\alpha_2)t^2+\ldots\ldots\ldots+(\alpha_{n-1}+\alpha_n)t^{n-1}+\alpha_n t^n$

$= 0.1+0.t+0.t^2+\ldots\ldots+0.t^{n-1}+0.t^n$

$\Rightarrow \quad \alpha_0=0, \alpha_0+\alpha_1=0, \alpha_1+\alpha_2=0, \ldots\ldots\alpha_{n-1}+\alpha_n=0, \alpha_n=0$

$\Rightarrow \quad \alpha_0=\alpha_1=\alpha_2=\ldots\ldots=\alpha_{n-1}=\alpha_n=0$

Thus S is L.I.

(II) Let $\alpha_0+\alpha_1 t+\ldots\ldots+\alpha_n t^n$ be any polynomial is V_n **(R)**.

This is not a linear combination of the elements of S.

Thus L (S) $\neq$ V **(R)**.

Hence S is not a basis set of V **(R)** = **P**.

(*iii*) Here $S = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3, \ldots\ldots, 1+t+t^2+\ldots\ldots+t^{n-1}+t^n\}$

S is L.I.

Let $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in \mathbf{R}$ such that

$\alpha_0 . 1+\alpha_1(1+t)+\alpha_2(1+t+t^2)+\ldots\ldots+\alpha_n(1+t+t^2+\ldots\ldots+t^{n-1}+t^n)$

$= 0$ (zero polynomial)

$\Rightarrow \quad (\alpha_0+\alpha_1+\alpha_2+\ldots\ldots+\alpha_n).1+(\alpha_1+\alpha_2+\ldots\ldots+\alpha_n)t$

$+(\alpha_2+\alpha_3+\ldots\ldots+\alpha_n)t^2+\ldots\ldots+(\alpha_{n-1}+\alpha_n)t^{n-1}+\alpha_n t^n$

$= 0.1+0.t+0.t^2+\ldots\ldots+0.t^{n-1}+0.t^n$

$\Rightarrow \quad \alpha_0+\alpha_1+\alpha_2+\ldots\ldots+\alpha_n=0, \alpha_1+\alpha_2+\ldots\ldots+\alpha_n=0, \ldots\ldots, \alpha_{n-1}+\alpha_n=0, \alpha_n=0$

$\Rightarrow \quad \alpha_0=\alpha_1=\alpha_2=\ldots\ldots=\alpha_{n-1}=0$

Thus S is L.I.

L (S) = V (P).

Let $a_0+a_1 t+\ldots\ldots+a_n t^n$ be any polynomial in V_n **(R)**.

Then $a_0+a_1 t+\ldots\ldots+a_n t^n = a_n t^n+a_{n-1}t^{n-1}+\ldots\ldots+a_1 t+a_0$

$= a_n(1+t+t^2+\ldots\ldots+t^{n-1}+t^n)+(a_{n+1}-a_n)(1+t+t^2+\ldots\ldots+t^{n-1})+$

$\ldots\ldots+(a_1-a_2)(1+t)+(a_0-a_1)(1)$

$\Rightarrow$ it is a linear combination of elements of S

Thus L (S) = V **(R)**.

Hence S is a basis set of V **(R)** = $\mathbf{P}_n$.

Example 12. *Determine whether or not each of the following forms a basis of R^2 :*

(*i*) (*1, 1*), (*3, 1*) (*ii*) (*0, 1*), (*0, – 3*) (*iii*) (*2, 1*), (*1, – 1*), (*0, 2*).

Sol. (*i*) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-3)]$$

Thus the two rows of A are L.I.

$\Rightarrow$ the vectors are L.I.

Hence these form a basis.

(*ii*) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{*} \qquad [\textit{Operating } R_{21}(3)]$$

*** [This form is called echelon form]**

Thus the echelon matrix has a zero row, *i.e.*, has one non-zero row

⇒ the vectors are L.D.

Hence these do not form a basis.

(*iii*) is not the basis of R^2 because a basis of R^2 must contain exactly two vectors.

Example 13. *Determine whether or not each of the following forms a basis of R^3* :

(*i*) $(1, 1, 1), (1, -1, 5)$

(*ii*) $(1, 2, -1), (0, 3, 1)$ (P.U. 1995)

(*iii*) $(2, 4, -3), (0, 1, 1), (0, 1, -1)$ (*iv*) $(1, 1, 1), (1, 2, 3), (2, -1, 1)$

(*v*) $(1, 3, -4), (1, 4, -3), (2, 3, -11)$ (*vi*) $(1, 1, 2), (1, 2, 5), (5, 3, 4)$

(*vii*) $(1, 5, -6), (2, 1, 8), (3, -1, 4), (2, 1, 1)$ (P.U. 1998)

(*viii*) $(1, 1, 1), (1, 0, -1), (3, -1, 0), (2, 1, -2)$.

Sol. (*i*) and (*ii*) are not bases of R^3 because a basis of R^3 must contain exactly three vectors.

(*iii*) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors.

$$A = \begin{bmatrix} 2 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad [\textit{Operating } C_{23}(1)]$$

Thus all the three rows of A are L.I.

⇒ the vectors are L.I.

Hence these form a basis.

(*iv*) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-1), R_{31}(-2)]$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad [\textit{Operating } C_{23}(-3)]$$

Thus all the three rows of A are L.I.

$\Rightarrow$ the vectors are L.I.

Hence these form a basis.

(v) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -3 \\ 2 & 3 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-1), R_{31}(-2)]$$

$$\sim \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{33}(3)]$$

Thus the echelon matrix has a zero row, *i.e.*, has two non-zero rows

$\Rightarrow$ the vectors are L.D.

Hence these do not form a basis.

(vi) The vectors form a basis iff they are L.I.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-1), R_{31}(-5)]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{32}(2)]$$

Thus the echelon matrix has zero row, *i.e.*, has two non-zero rows

$\Rightarrow$ the vectors are L.D.

Hence these do not form a basis.

(vii) and ($viii$) are not bases of R^3 because a basis of R^3 must contain exactly three vectors.

Example 14. *Under what conditions on the scalar a do the vectors* $(1, 1, 1)$ *and* $(1, a, a^2)$ *form a basis of* V_3 **(C)** ?

Sol. Since V_3 **(C)** is a 3-dimensional vector space,

$\therefore$ the basis of V_3 **(C)** will contain exactly three vectors.

Hence the set $S = \{(1, 1, 1), (1, a, a^2)\}$ cannot be the basis of V_3 **(C)** and so no condition on a can be found out.

Example 15. *Let W be the space generated by the polynomials*

$$v_1 = t^3 - 2t^2 + 4t + 1, \quad v_2 = 2t^3 - 3t^2 + 9t - 1,$$
$$v_3 = t^3 + 6t - 5, \qquad v_4 = 2t^3 - 5t^2 + 7t + 5.$$

Find a basis and the dimension of W. (G.N.D.U. 1986 S)

Sol. The co-ordinate vectors of the given polynomial relative to be basis $\{t^3, t^2, t, 1\}$ are

$[v_1] = (1, -2, 4, 1), [v_2] = (2, -3, 9, -1),$

$[v_3] = (1, 0, 6, -5), [v_4] = (2, -5, 7, 5)$ respectively.

Thus form the matrix whose rows are given vectors

$$A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad \text{[Operating } R_{21}(-2), R_{31}(-1), R_{41}(-2)]$$

$$\sim \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{[Operating } R_{32}(-2), R_{42}(1)]$$

The non-zero rows $(1, -2, 4, 1)$ and $(0, 1, 1, -3)$ of the above echelon matrix form the basis.
Hence the corresponding polynomials $t^3 - 2t^2 + 4t + 1$ and $t^2 + t - 3$ form a basis of W.
Hence dim. W = 2.

Example 16. *If W be the sub-space generated by the polynomials*

(*a*) $x = t^3 + 2t^2 - 2t + 1,\ \ y = t^3 + 3t^2 - t + 4,\ z = 2t^3 + t^2 - 7t - 7.$ (G.N.D.U. 1988 S, 87)

(*b*) $v_1 = t^3 - 2t^2 + 4t + 1,\ v_2 = 2t^3 - 3t^2 + 9t - 1,$
$v_3 = t^3 + 6t - 5,\qquad v_4 = 2t^3 - 5t^2 + 7t + 5.$ (Pbi. U. 1986)

Find a basis and dimension of W.

Sol. (*a*) The co-ordinate vectors of the given polynomials relative to the basis $\{t^3, t^2, t, 1\}$ are
$[x] = (1, 2, -2, 1), [y] = (1, 3, -1, 4),$
$[z] = (2, 1, -7, -7)$ respectively.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix} \qquad \text{[Operating } R_{21}(-1), R_{32}(-3)]$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[Operating } R_{32}(-3)]$$

The non-zero rows $(1, 2, -2, 1)$ and $(0, 1, 1, 3)$ of the above echelon matrix form the basis.
Hence the corresponding polynomials $t^3 + 2t^2 - 2t + 1$ and $t^2 + t + 3$ form a basis of W.
Hence dim. W = 2.

(*b*) The co-ordinate vectors of the given polynomials relative to the basis $\{t^3, t^2, t, 1\}$ are
$[v_1] = (1, -2, 4, 1),\ [v_2] = (2, -3, 9, -1),$
$[v_3] = (1, 0, 6, -5),\ [v_4] = (2, -5, 7, 5)$ respectively.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -2 & -2 & 6 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2), R_{31}(-1), R_{42}(-1)]$$

$$\sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{32}(-2) \textit{ and } R_{43}(1)]$$

Thus non-zero rows $(1, -2, 4, 1)$ and $(0, 1, 1, -3)$ of the above echelon matrix form the basis.
Hence the corresponding polynomials $t^3 - 2t^2 + 4t + 1$ and $t^2 + t - 3$ form a basis of W.
Hence dim W = 2.

Example 17. *Find the dimension of the sub-space W of R^4 generated by*
(*i*) $(1, 4, -1, 3), (2, 1, -3, -1)$ *and* $(0, 2, 1, -5)$
(*ii*) $(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)$. (G.N.D.U. 1987 S, 85)

Sol. (*i*) Let the given co-ordinate vectors be

$[x] = (1, 4, -1, 3)$, $[y] = (2, 1, -3, -1)$ and $[z] = (0, 2, 1, -5)$.

Thus form the matrix whose rows are given vectors

$$A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 1 & -3 & -1 \\ 0 & 2 & 1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & -7 & -1 & -7 \\ 0 & 2 & 1 & -5 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2)]$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & -7 & -1 & -7 \\ 0 & -5 & 0 & -12 \end{bmatrix} \qquad [\textit{Operating } R_{32}(1)]$$

The non-zero rows $(1, 4, -1, 3)$, $(0, -7, -1, -7)$ and $(0, -5, 0, -12)$ form the basis.

Hence dim. W = 3.

(*ii*) **Exactly similar to part (*i*).** **[Ans.** dim. W = 2]

Example 18. *Let V be a space of 2×2 matrices over R and let W be the sub-space generated by*

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \textit{ and } \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}.$$

Show that

(*i*) $\left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ *forms a basis set*

(*ii*) *dim.* $W = 2$.

Sol. The basis set of V (**R**) is

$$S_1 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

The co-ordinates of vectors x_1, x_2, x_3, x_4 relative to the basis S_1 are

$(1, -5, -4, 2), (1, 1, -1, 5), (2, -4, -5, 7), (1, -7, -5, 1)$ respectively.

Thus form the matrix whose rows are given vectors

$$A \cdot = \begin{pmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{pmatrix} \qquad [\text{Operating } R_{21}(-1), R_{31}(-2), R_{41}(-1)]$$

$$\sim \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad [\text{Operating } R_{32}(-1), R_{42}(1/3)]$$

The non-zero rows $(1, -5, -4, 2)$ and $(0, 6, 3, 3)$ of the above echelon matrix form the basis. Hence the set of corresponding matrices is

$$S_1 = \left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}\right\}, \text{ which forms a basis set W.}$$

Hence dim. $W = 2$.

Example 19. *Let V_1 and V_2 be the sub-spaces of R^4 generated by*

$\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$ *and*

$\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$ *respectively.*

Find the dimensions of

(*i*) V_1 (*ii*) V_2 (*iii*) $V_1 + V_2$ (*iv*) $V_1 \cap V_2$.

Sol. (*i*) Form a matrix A whose rows are the given vectors, and further reduce to echelon form

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \qquad [\text{Operating } R_{21}(-1), R_{31}(-2)]$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\text{Operating } R_{32}(-1)]$$

which is of echelon form.

Thus the basis of V_1 is $\{(1, 1, 0, -1), (0, 1, 3, 1)\}$

and hence dim. $V_1 = 2$.

(*ii*) Proceed as in part (*i*).

The basis of V_1 is $\{(1, 2, 2, -2), (0, -1, -2, 1)\}$

and dim. $V_2 = 2$.

(*iii*) $V_1 + V_2$ is the space generated by the vectors of V_1 and V_2 and consequently generated by bases of V_1 and V_2.

Thus we have

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad [\textit{Operating } R_{31}(-1), R_{42}(-1)]$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad [\textit{Operating } R_{32}(-1)]$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad [\textit{Operating } R_{43}(1)]$$

which is of echelon form

$\therefore$ dim. $(V_1 + V_2)$ = No. of non-zero rows = 3

(*iv*) dim. $(V_1 \cap V_2)$ = dim. V_1 + dim. V_2 – dim. $(V_1 + V_2) = 2 + 2 - 3 = 1$.

Example 20. *Let V_1 and V_2 be subspaces of R^4 given by*

$$V_1 = \{(a, b, c, d) \mid b - 2c + d = 0\}$$

and $$V_2 = \{(a, b, c, d) \mid a = d, b = 2c\}.$$

Find a basis and dimension of

(*i*) V_1 (*ii*) V_2 (*iii*) $V_1 \cap V_2$. (P.U. 1997 ; G.N.D.U. 1988, 85)

Sol. (*i*) $V_1 = \{(a, b, c, d) \mid b - 2c + d = 0\}$

The vectors of V_1 are of the type.

$(a, 2c - d, c, d)$ for all $a, c, d \in R$

These form a subspace of R^4 [*Do it*]

and $(a, 2c - d, c, d) = a(1, 0, 0, 0) + c(0, 2, 1, 0) + d(0, -1, 0, 1)$

$\Rightarrow$ each vector in V_1 is a linear combination of the vectors

$(1, 0, 0, 0), (0, 2, 1, 0)$ and $(0, -1, 0, 1)$.

These vectors of V_1 are L.I. over R [*Justify* !]

and as such these form a basis of V_1 and dim. $V_1 = 3$.

(*ii*) $V_2 = \{ (a, b, c, d) \mid a = d, b = 2c\}$.

The vectors of V_2 are of the type

$(a, 2c, c, a)$ for all $a, c \in R$.

These form a subspace of R^4 *[Do it]*

and $(a, 2c, c, a) = a(1, 0, 0, 1) + c(0, 2, 1, 0)$

$\Rightarrow$ each vector in V_2 is a linear combination of the vectors

$(1, 0, 0, 1)$ and $(0, 2, 1, 0)$.

These vectors of V_2 are L.I. over R *[Justify !]*

and as such these form a basis of V_2 and dim. $V_2 = 2$.

(*iii*) $V_1 \cap V_2 = \{(a, b, c, d) \mid b - 2c + d = 0, a = d, b = 2c\}$.

Now $b - 2c + d = 0, a = d, b = 2c \Rightarrow a = 0, d = 0$ and $b = 2c$

$\therefore$ The vectors of $V_1 \cap V_2$ are of the type

$(0, 2c, c, 0)$ for all $c \in R$.

These form a subspace of R^4 *[Do it]*

and $(0, 2c, c, 0) = c(0, 2, 1, 0)$

$\Rightarrow$ each vector in $V_1 \cap V_2$ is a linear combination of $(0, 2, 1, 0)$.

These vectors of $V_1 \cap V_2$ are L.I. over R. *[Justify !]*

Hence basis of $V_1 \cap V_2 = \{0, 2, 1, 0\}$ and dim. $(V_1 \cap V_2) = 1$.

11. Echelon Matrices

(*a*) **Echelon Matrix. Def.** *A matrix $A = [a_{ij}]_{m \times n}$ is said to be an echelon matrix if the number of zeros after non-zero elements increases row by row ; the last row may consist of all zeros.*

For examples :

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}\text{; etc. are all matrices in the echelon form.}$$

First non-zero elements in the row are called **distinguished elements.**

(*b*) **Row-Reduced Echelon Matrix. Def.** *An echelon matrix ; say A, is said to be a row-reduced echelon matrix if:*

(*i*) *the non-zero rows of A are at the top,*

(*ii*) *the distinguished elements are each equal to 1,*

(*iii*) *the distinguished elements are the only non-zero elements in their respective columns.*

For examples :

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are row reduced echelon matrices.

12. Row and Column Space

(*i*) **Row Equivalence of two matrices. Def.** *Let A and B be two matrices. The matrix A is said to be* **row equivalent** *to the matrix B if B can be obtained from A by a finite number of elementary row transformations.*

(*ii*) **Column Equivalence of two matrices. Def** *Let A and B be two matrices. The matrix A is said to be column equivalent to the matrix B if B can be obtained from A by a finite number of elementary column transformations.*

13. Elementary Transformations

(I) Interchanging of ith and jth rows $(R_i \leftrightarrow R_j)$

Interchanging of ith and jth columns. $(C_i \leftrightarrow C_j)$

(II) Multiplying ith row by $k\ (\neq 0)$ $(R_i \rightarrow k\,R_i, k \neq 0)$

Multiplying jth column by $k\ (\neq 0)$. $(C_j \rightarrow k\,C_j, k \neq 0)$

(III) Adding to the ith row, k times the jth row $(R_i \rightarrow R_i + k\,R_j, k \neq 0)$

Adding to the ith column, k times the jth column $(C_i \rightarrow C_i + k\,C_j, k \neq 0)$

14. Row Space of a Matrix

Def. Let $A = [a_{ij}]$ be any $m \times n$ matrix over the field F. Rows of A, each having n co-ordinates, are

$$R_1 = (a_{11}, a_{12}, \ldots\ldots, a_{1n})$$
$$R_2 = (a_{21}, a_{22}, \ldots\ldots, a_{2n})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$R_m = (a_{m1}, a_{m2}, \ldots\ldots, a_{mn}).$$

These are the numbers of F^n.

The linear span of these row vectors, i.e., $L\,[(R_1, R_2, \ldots\ldots, R_m\}]$ *is said to be row-space of A and is a sub-space of* F^n.

The **row space** of A = **L** $[\{R_1, R_2, \ldots\ldots, R_m\}]$.

Similarly **column space** of A = L $[\{C_1, C_2, \ldots\ldots, C_n\}]$,

where columns of A, each having m co-ordinates, are

$$C_1 = (a_{11}, a_{21}, \ldots\ldots, a_m)$$
$$C_2 = (a_{12}, a_{22}, \ldots\ldots, a_{m2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$C_n = (a_{1n}, a_{2n}, \ldots\ldots, a_{mn}).$$

THEOREMS

Theorem I. *Row equivalent matrices have same row space.*

Proof. Let A and B be two row-equivalent matrices.

By def., each row of B is either that of A or is a linear combination of rows of A

$\Rightarrow$ row space of A contains row space of B ...(1)

If we proceed from B to A by inverse elementary row operation we obtain that the row space of B contains row space of A. ...(2)

From (1) and (2), row space of A = row space of B.

Theorem II. *Row reduced echelon matrices have the same row space iff they have same non-zero rows.*

Proof is omitted.

15. Row and Column Spaces and Rank of a Matrix

Let A be any $m \times n$ matrix.

Let $R_1, R_2, \ldots\ldots, R_m$ and $C_1, C_2, \ldots\ldots, C_n$ be the rows and columns of A respectively.

Then R_i is a row *n-vector* and C_j is a column *m-vector*.

(I) **Row Space.** *The sub-space of V_n spanned by the rows $R_1, R_2, \ldots\ldots, R_m$ is called the row-space of A.*

(II) **Column Space.** *The sub-space of V_n spanned by the columns $C_1, C_2, \ldots\ldots, C_n$ is called the column space of A.*

(III) **Column Rank.** *The dimension of the row-space of a matrix A is called the row-rank of A.*

(IV) **Column Rank.** *The dimension of the column-space of a matrix A is called the column-rank of A.*

(P.U. 1989)

THEOREM

Invariance of Row-Rank and Column-Rank

(*a*) *Pre-multiplication by a non-singular matrix does not change the row-rank of a matrix.*

(*b*) *Post-multiplication by a non-singular matrix does not change the column-rank of a matrix.*

Proof. (*a*) Let A be a $m \times n$ matrix and P a non-singular $m \times n$ matrix.

$$\text{Then} \quad B = PA = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & R_{mm} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{pmatrix}$$

$$\Rightarrow \quad B = \begin{pmatrix} p_{11} R_1 + p_{12} R_2 + \ldots\ldots + p_{1m} R_m \\ p_{21} R_1 + p_{22} R_2 + \ldots\ldots + p_{2m} R_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{m1} R_1 + p_{m2} R_2 + \ldots\ldots + p_{mm} R_m \end{pmatrix}$$

Since each row of B is a linear combination of the rows of A,

$\therefore$ row space of B is a subset of the row-space of A ...(1)

Now $B = PA \Rightarrow A = P^{-1} B$ [$\because$ *P is non-singular,* $\therefore$ $P \neq 0$]

$\therefore$ As above, the row space of A is a subset of the row-space of B.

From (1) and (2), row-space of A = row-space of B

$\Rightarrow$ row-rank of B = row-rank of A ...(2)

Hence the result.

(*b*) **Please try yourself.**

THEOREMS

Theorem I. *If r be the row-rank of $m \times n$ matrix A, then there exists a non-singular matrix R such that* $RA = \begin{bmatrix} G \\ O \end{bmatrix}$, *where G is an $r \times n$ matrix consisting of a set of r linearly independent rows of A.*

(G.N.D.U. 1987)

Proof. Since r is the row rank of A,

$\therefore$ A has r linearly independent rows.

Now we bring these rows to first r places with the help of elementary row transformations of A.

The last $(m - r)$ rows, being linearly dependent, are linear combinations of the first r rows and thus they can be made zero by proper row-operations without affecting the first r rows.

Also we know that row-transformation can be affected by pre-multiplying with a non-singular matrix.

Let R be the product of such non-singular pre-multipliers.

$\therefore$ $\quad RA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is $r \times n$ matrix having a set of r linearly independent rows of A.

Theorem II. *If s be the column rank of an $m \times n$ matrix of A, then there exists a non-singular matrix R such that AR = [H O], where H is an $m \times n$ matrix having a set of s linearly independent columns of A.*

Proof. Please try yourself.

16. Equality of Row Rank, Column Rank and Rank

Theorem I. (*a*) *The row rank of a matrix is the same as its rank.*

(*b*) *The column rank of a matrix is the same as its rank.*

Proof. (*a*) Let A be any $m \times n$ matrix.

Let r_1 be the rank of A and r_2 be the row-rank of A.

To prove. $\quad r_1 = r_2$.

Since rank of $A = r$,

$\therefore$ there exists a non-singular matrix R such that

$RA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is $r \times n$ matrix. [*Above Th. I*]

Now row rank of RA = row rank of A = r_1

$\therefore$ $\begin{bmatrix} G \\ O \end{bmatrix}$ has at the most r_1 linearly independent rows

$\therefore$ row rank of $\begin{bmatrix} G \\ O \end{bmatrix}$ is at the most r_1

$\Rightarrow$ $\quad r_2 \le r_1$

Again since row-rank of A = r_2, ...(1)

$\therefore$ there exists a non-singular matrix T, such that

$TA = \begin{bmatrix} H \\ O \end{bmatrix}$, where H is $r_2 \times n$ matrix.

Now ρ (TA) = ρ (A) = r_1 and every minor of order $(r_2 + 1)$ of $\begin{bmatrix} H \\ O \end{bmatrix} = 0$.

$\therefore$ rank of $\begin{bmatrix} H \\ O \end{bmatrix}$ is $\le r_2$

$\Rightarrow$ $\quad r_1 \le r_2$ ...(2)

From (1) and (2), we have $r_1 = r_2$, which proves the theorem.

(*b*) Since columns of A are rows of A',

$\therefore$ Column rank of A = row rank of A'

= rank of A' [*Th. I* (*a*)]

= rank of A

Hence the theorem.

Cor. *The row-rank of a matrix is equal to its column rank.* (G.N.D.U. 1993, 88)

Proof. Combining the results of Theorem I (*a*) and I (*b*), we get the required result.

Theorem II. $\rho(A+B) \le \rho(A) + \rho(B)$, *where A and B are* $m \times n$ *matrices.*

Proof. Let S (A) denote the row-space of A,

S (B) denote the row-space of B

and S (A + B) denote the row-space of A + B,

where S is the sub-space spanned by the rows of A and B.

Since the number of members in a basis ≤ the sum of the number of members in the basis of A and the basis of B.

$\therefore$ dim. S ≤ dim. S (A) + dim. S (B) ...(1),

where dim. S denotes the dimensions of the space S ; etc.

Also row-space S (A + B) is a sub-space of S

$\therefore$ dim. S (A + B) ≤ dim. (S) ...(2),

From (1) and (2), we have dim. S (A + B) ≤ dim. S (A) + dim. S (B).

Hence $\rho(A+B) \le \rho(A) + \rho(B)$.

Theorem III. *If A, B are two n-rowed matrices, then* $\rho(AB) \ge \rho(A) + \rho(B) - n$.

Proof. Let $\rho(A) = r$.

Then there exist non-singular matrices R and T, such that

$$RAT = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

$$\therefore \qquad A = R^{-1} \begin{bmatrix} I_r & O \\ O & I_{n-r} \end{bmatrix} T^{-1} \qquad \text{...(1),}$$

where null matrices O, on R.H.S., have suitable orders.

We define another n-rowed square matrix C, such that

$$C = R^{-1} \begin{bmatrix} O & O \\ O & I_{n-r} \end{bmatrix} T^{-1} \qquad \text{...(2),}$$

where null matrices O, on R.H.S., have suitable orders.

From (1) and (2),

$$A + C = R^{-1} \begin{bmatrix} I_r & O \\ O & I_{n-r} \end{bmatrix} T^{-1} = R^{-1} I_n T^{-1} = R^{-1} T^{-1}.$$

Since R and T are non-singular, then R^{-1} and T^{-1} are also non-singular.

$\therefore$ A + C is also non-singular and of order $n \times n$

$\therefore$ $\rho(A+C) = n$.

Also $\rho(C) = n - r = n - \rho(A)$ ...(3)

Now $\rho(B) = \rho[(A+C)B]$ [$\because$ *A + C is non-singular*]

$= \rho(AB + CB)$

$\le \rho(AB) + \rho(CB)$ ...(4) [*Th. II*]

Also $\rho(CB) \le \rho(C)$ $[\because \rho(AB) \le \rho(A)]$

$= n - \rho(A)$ [*Using (3)*]

$\therefore$ from (4), $\rho(B) \le \rho(AB) + n - \rho(A)$.

Hence $\rho(AB) \ge \rho(A) + \rho(B) - n$.

SOLVED EXAMPLES

Example 1. *Reduce the following to echelon form and then to row reduced form :*

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}.$$

Sol.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2) \textit{ and } R_{31}(-3)]$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{bmatrix}, \qquad [\textit{Operating } R_3 \to 3R_3 + (-7)R_2]$$

which is of echelon form.

Now

$$A \sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \qquad \left[\textit{Operating } R_2 \to \frac{1}{3}R_2 \textit{ and } R_3 \to \frac{1}{7}R_3\right]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \qquad [\textit{Operating } R_{12}(2)]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix}, \qquad \left[\textit{Operating } R_{23}\left(\frac{4}{3}\right) \textit{ and } R_{13}\left(-\frac{1}{3}\right)\right]$$

which is of row-reduced echelon form.

Example 2. (*a*) *Reduce the matrix*

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$$

to row reduced form. Also find a basis for the row space and its dimension. (P.U. 1985 S)

(*b*) *Find a basis for the row space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 4 & 7 & 11 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

and hence find the dimension of the row space of the matrix A. (P.U. 1985)

Sol. (*a*)

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix} \qquad [\textit{Operating } R_{32}(-2)]$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 0 & 3 & -9 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{31}(-3)]$$

$$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} & 2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\textit{Operating } R_1 \to \frac{1}{2} R_1\right]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \left[\textit{Operating } R_{12}\left(\frac{1}{2}\right)\right]$$

which is of row reduced echelon form.

$$\text{Basis for row space} = \left\{\left(1, 0, \frac{1}{2}, -\frac{1}{2}\right), (0, 1, -3, -1)\right\}$$

and dimension of row space = 2.

(*b*)

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 4 & 7 & 11 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \end{bmatrix} \quad \textit{[Operating } R_{21}(-1), R_{32}(-1) \textit{ and } R_{41}(-1)\textit{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \end{bmatrix} \quad \textit{[Operating } R_{21}(-1) \textit{ and } R_{32}(-1)\textit{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{[Operating } R_{42}(-1)\textit{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \textit{[Operating } R_2 \to (-1) R_2\textit{]}$$

which is of row reduced echelon form.

Basis for row space = {(1, 2, 3, 5), (0, 1, 1, 2)}

and dimension of row space = 2.

Example 3. *Find the rank of the matrix*

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}. \quad \text{(G.N.D.U. 1998)}$$

Sol. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix} \quad \textit{[Operating } R_2 \to R_2 - 2R_1 \textit{ and } R_3 \to R_3 - 3R_1\textit{]}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}, \quad \textit{[Operating } R_{32}(-1)\textit{]}$$

which is an echelon form of A, having two non-zero rows.

Hence rank of A = Row rank of A = 2.

Example 4. *Find the column ranks of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$

Now column rank of A = row rank of A$'$, where

Now try yourself.

$$A' = \begin{bmatrix} 1 & 4 & 5 & -1 \\ 1 & 5 & 8 & -2 \\ 2 & 5 & 1 & 2 \end{bmatrix}.$$

Now try yourself. **[Ans. 3]**

(*ii*) **Please try yourself** **[Ans. 2]**

Example 5. *Find the column rank of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 3 \\ 3 & 1 & -1 & -4 \\ -1 & -2 & -3 & -7 \\ -4 & -3 & -2 & -8 \end{bmatrix}. \qquad \text{(P.U. 1989)}$$

Sol. We know that column rank of A is the same as the row rank of A$'$.

Now $A' = \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 1 & 3 & 1 & -2 & -3 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & -4 & -7 & -8 \end{bmatrix}$ [*Operating* $R_{21}(-1)$, $R_{31}(-1)$, $R_{41}(-2)$]

$$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & -2 & -4 & -2 & 2 \\ 0 & -5 & -10 & -5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_{32}(-2), R_{42}(-5)]$$

which is in the echelon form.

Since A$'$ (in its echelon form) contains two non-zero rows,

$\therefore$ row rank of A$'$ = 2

Hence column rank of A = 2.

Example 6. *Find the rank, row rank and column rank of the matrix*

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$$

and show that rank, row rank and column rank are same. (P.U. 1992)

Sol. Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$ [*Operating* $R_{21}(-2)$ *and* $R_{31}(-3)$]

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ [*Operating* $R_{32}(-2)$]

Since the echelon matrix of the given matrix has two non-zero rows,

$\therefore$ rank of the given matrix = 2.

Similarly find row and column ranks, each = 2. [*Do it*]

Hence the result.

Example 7. *Determine whether the following matrices have the same row space :*

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}.$$ (G.N.D.U. 1987 S)

Sol. $A = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$ [*Operating* $R_{21}(-3)$]

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$ ...(1) [*Operating* $R_{12}(-1)$]

$$B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$ [*Operating* $R_{31}(-3)$]

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$ ...(2) [*Operating* $R_{12}(-1)$]

and $C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$$ [*Operating* $R_{21}(-4)$, $R_{31}(-3)$]

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ [*Operating* $R_{32}(-2)$]

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{...(3)} \quad [\textit{Operating } R_{32}(-2)]$$

Since non-zero rows of reduced forms of B and C are same,

$\therefore$ row space of B = row space of C.

But non-zero row of reduced form of A is different from those of B and C.

Hence A has different row space.

Example 8. *Determine whether the following matrices have the same column space :*

(*a*) $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{bmatrix}$

(*b*) $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{bmatrix}$ (Pbi. U. 1996 ; G.N.D.U. 1992 S)

Sol. A and B have the same column space iff A^t and B^t have same row space.

(*a*) $A^t = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2), R_{31}(-3)]$$

$$\sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{32}(-2)]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{...(1)} \quad [\textit{Operating } R_{12}(2)]$$

And $B^t = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-3), R_{31}(-5)]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{12}(2)]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{...(2)} \quad [\textit{Operating } R_{12}(-1)]$$

From (1) and (2), non-zero rows of reduced form of A' and B' are same.

$\therefore$ row space of A' = row space of B'.

Hence column space of A = column space of B.

(*b*) $A' = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}$ [*Operating $R_{21}(-3)$, $R_{31}(-5)$*]

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ [*Operating $R_{32}(2)$*]

$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ ...(1) [*Operating $C_{21}(-1)$ and $C_{32}(2)$*]

And $B' = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}$ [*Operating $R_{21}(-2)$ and $R_{31}(-3)$*]

$\sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ [*Operating $R_{32}(-2)$*]

$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ ...(2) [*Operating $R_{12}(2)$*]

From (1) and (2), non-zero rows of reduced form of A' and B' are same.

$\therefore$ row space of A' = row space of B'.

Hence column space of A = column space B.

Example 9. *Show that the space U generated by the vectors*

$$u_1 = (1, 2, -1, 3)\,;\ u_2 = (2, 4, 1, -2),\ u_3 = (3, 6, 3, -7)$$

and the space V generated by the vectors

$$v_1 = (1, 2, -4, 11),\ v = (2, 4, -5, 14)$$

are equal, i.e., U = V. (P.U. 1986)

Sol. Form a matrix A whose rows are vectors u_1, u_2 and u_3 and reduce it to echelon form.

$\therefore$ $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2), R_{31}(-3)]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\textit{Operating } R_{32}(-2)]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \ldots(1) \quad [\textit{Operating } \tfrac{1}{3} R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \ldots(2)$$

Now form the matrix B whose rows are υ_1 and υ_2 and reduce it to echelon form.

$$\therefore \quad B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \qquad [\textit{Operating } R_1 \to R_1 + R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \qquad [\textit{Operating } R_{21}(-2)]$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} \qquad [\textit{Operating } 1/3\, R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} \qquad \ldots(2) \quad [\textit{Operating } R_{12}(4)]$$

From (1) and (2), we see that the non-zero rows in the reduced echelon matrices are identical, and therefore, row spaces of A and B are equal and thus U = V.

Also, basis of each $= \left\{ \left(1, 2, 0, \frac{1}{3}\right), \left(0, 0, 1, -\frac{8}{3}\right) \right\}$.

Example 10. *Let A and B be arbitrary $m \times n$ matrices, prove that*

(i) $\rho(AB) \le \rho(A)$ *(ii)* $\rho(AB) \le \rho(B)$.

Sol. Let A and B be $m \times n$ and $n \times p$ matrices respectively.

Let $\rho(A) = r_1$, $\rho(B) = r_2$ and $\rho(AB) = r$.

(*i*) There exists a non-singular matrix P such that

$$PA = \begin{bmatrix} G \\ O \end{bmatrix},$$

where G is an $r_1 \times n$ matrix of rank r_1.

Also as the rank of a matrix does not alter on multiplication with non-singular matrix,

$$\therefore \quad r = \rho(AB) = \rho(PAB) \qquad [\because \textit{ P is non-singular}]$$

$$= \text{rank of} \begin{bmatrix} G \\ O \end{bmatrix} B \qquad \ldots(1)$$

Further as G has at the most r_1 non-zero rows,

$\therefore$ $\begin{bmatrix} G \\ O \end{bmatrix}$ B cannot have more than r_1 non-zero rows which may arise due to the multiplication of r_1 non-zero rows of G and columns of B

$\therefore$ rank cf $\begin{bmatrix} G \\ O \end{bmatrix} B \le r_1$ ...(2)

From (1) and (2), $r \le r_1$.

Hence $\rho(AB) \le \rho(A)$. ...(3)

(*ii*) $r = \rho(AB) = \rho[(AB)^t]$ $[\because \rho(A) = \rho(A^t)]$

$= \rho(B^t A^t) \le \rho(B^t)$ [*Using* (3)]

$= \rho(B)$ $[\because \rho(B^t) = \rho(B)]$

Hence $\rho(AB) \le \rho(B)$.

Example 11. *Give examples of 2 × 2 matrices A and B such that*

(*i*) $\rho(A+B) < \rho(A), \rho(B)$ (*ii*) $\rho(A+B) = \rho(A) = \rho(B)$

(*iii*) $\rho(A+B) > \rho(A), \rho(B)$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

$$\therefore \quad A + B = \begin{bmatrix} 1-1 & 1-1 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore$ $\rho(A+B) = 0$.

and $\rho(A) = \rho(B) = 1$.

Hence $\rho(A+B) < \rho(A), \rho(B)$.

(*ii*) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$$\therefore \quad A + B = \begin{bmatrix} 1+0 & 0+2 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A+B) = 1$

Also $\rho(A) = 1 = \rho(B)$.

Hence $\rho(A+B) = \rho(A) = \rho(B)$.

(*iii*) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad A + B = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\det(A+B) = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

$\therefore$ $\rho(A+B) = 2$

But $\rho(A) = 1$ and $\rho(B) = 1$

Hence $\rho(A+B) > \rho(A), \rho(B)$.

Example 12. *Prove that if $m \times n$ matrix satisfies the equation $A^2 = A$, then $\rho(A) + \rho(I_n - A) = n$.*

Sol. We have

$$A^2 = A \Rightarrow A - A^2 = O \Rightarrow A(I_n - A) = O$$

Since $\quad A + (I_n - A) = I_n$,

$\therefore \quad \rho(A + I_n - A) \le \rho(A) + \rho(I_n - A)$

$\Rightarrow \quad \rho(I_n) \le \rho(A) + \rho(I_n - A)$

$\Rightarrow \quad n \le \rho(A) + \rho(I_n - A) \qquad \ldots(1)$

Again since $A(I_n - A) = O$,

$\therefore \quad \rho(A) + \rho(I_n - A) - n \le \rho(O)$

$\Rightarrow \quad \rho(A) + \rho(I_n - A) - n \le 0$

$\Rightarrow \quad \rho(A) + \rho(I_n - A) \le n \qquad \ldots(2)$

From (1) and (2), $\rho(A) + \rho(I_n - A) = n$.

17. Application to Linear Equations

Let us consider a system of m linear equations in n unknowns ; say $x_1, x_2, \ldots, x_n$ over a field F :

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(A)$$

The system can be written as AX = B,

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$.

The augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & \ldots & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & \ldots & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & \ldots & b_m \end{bmatrix} \qquad \ldots(B)$$

The equations (A) are **dependent** if the rows of (B) are L.D. and **independent** if the rows of (B) are L.I.

Note. Two systems of linear equations are said to be equivalent if they have the same row space, *i.e.*, iff corresponding augmented matrices are row equivalent.

Theorem. *The system of equations*

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right.$$

has a solution iff

$\rho(A) = \rho(AB)$, *where A and B are as above.*

Proof. The given system of equations AX = B

i.e.,
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

This is equivalent to $x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ a_m \end{bmatrix}$

The system AX = B has a solution

iff the column vector of B is a linear combination of the columns of A

iff the column vector of B belongs to the column space of A

iff the column space of A = column rank of [AB].

iff the column rank of A = column rank of [AB].

Hence the given system AX = B has a solution iff ρ (A) = ρ (AB).

SOLVED EXAMPLES

Example 1. *Find the basis and the dimension of the solution space W of the following system of equations :*

(*i*)
$$\begin{aligned} x + 2y - 4z + 3s - t &= 0 \\ x + 2y - 2z + 2s + t &= 0 \\ 2x + 4y - 2z + 3s + 4t &= 0 \end{aligned}$$
(P.U. 1997 ; G.N.D.U. 1996, 92 S, 90)

(*ii*)
$$\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ x + 2y + 3z + s + t &= 0 \\ 3x + 6y + 8z + s + 5t &= 0 \end{aligned}$$
(P.U. 1998 ; Pbi. U. 1987 ; G.N.D.U. 1986)

(*iii*)
$$\begin{aligned} x + 2y - 2z + 2s - t &= 0 \\ x + 2y - z + 3s - 2t &= 0 \\ 2x + 4y - 7z + s + t &= 0 \end{aligned}$$
(G.N.D.U. 1992, 85 S)

(*iv*)
$$\begin{aligned} x + 2y - 2z + 2r + s &= 0 \\ 2x + 4y - 6z + 5r &= 0 \\ 2x + 4y - 2z + 3r + 4s &= 0 \\ 3x + 6y - 8z + 7r + s &= 0. \end{aligned}$$
(P.U. 1987)

Sol. (*i*) The given system can be written as AX = B,

where $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 1 & 2 & -2 & 2 & 1 \\ 2 & 4 & -2 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now $A \sim \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 6 & -3 & 6 \end{bmatrix}$ [*Operating* $R_{21}(-1)$, $R_{31}(-2)$]

$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ [*Operating* $R_{32}(-3)$]

$\sim \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ [*Operating* $R_{12}(2)$]

The given system reduces to

$$x + 2y + s + 3t = 0$$
$$2z - s + 2t = 0$$

i.e., $x = -2y - s - 3t$

$$z = \frac{1}{2}s - t$$

i.e., $x = -2y - s - 3t$

$$z = 0\,y + \frac{1}{2}s - t.$$

$\therefore$ The solution set is $\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} t.$

We have five unknowns and two non-zero equations of echelon form.

Thus the three variables y, s, t are independent and two variables x and z are dependent.

Hence the basis of solution space S = {(– 2, 1, 0, 0, 0), (– 1, 0, 1/2 , 1, 0), (– 3, 0, – 1, 0, 1)}

and dim. S = 3.

(*ii*) The given system can be written as AX = B,

where $A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $A \sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix}$ [*Operating* $R_{21}(-1)$ *and* $R_{31}(-3)$]

$\sim \begin{bmatrix} 1 & 2 & 0 & -5 & 7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ [*Operating* $R_{12}(-2)$ *and* $R_{32}(-2)$]

The given system reduces to

$$x + 2y - 5s + 7t = 0$$
$$z + 2s - 2t = 0$$

$\Rightarrow$ $x = -2y + 5s - 7t$

and $z = 0\,y - 2s + 2t$

$\therefore$ The solution set is $\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$

We have five unknowns and two non-zero equations of echelon form.

Thus the three variables y, s, t are independent and two variables x and z are dependent. Hence the basis of solution space S

$= \{(-2, 1, 0, 0, 0), (5, 0, -2, 1, 0), (-7, 0, 2, 0, 1)\}$ and dim. S = 3.

(*iii*) The given system can be written as AX = B,

where $A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $A \sim \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$ [*Operating* $R_{21}(-1)$ *and* $R_{31}(-2)$]

$\sim \begin{bmatrix} 1 & 2 & 0 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ [*Operating* $R_{12}(2)$ *and* $R_{32}(3)$]

The given system reduces to

$$x + 2y + 4s - 3t = 0$$
$$z + s - t = 0$$

$\Rightarrow$ $x = -2y - 4s + 3t$ and $z = -s + t$

$\therefore$ The solution set is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} t$

We have five unknowns and for non-zero equation of echelon form.

Thus the three variables y, s, t are independent and two variables x and z are dependent

Hence the basis of solution space

S = $\{(-2, 1, 0, 0, 0), (-4, 0, -1, 1, 0), (3, 0, 1, 0, 1)\}$ and dim. S = 3.

(*iv*) Exactly similar to parts (*i*) – (*iii*).

[**Ans.** Basis = $\{(-2, 1, 0, 0, 0), (-2, 0, 1, 2, 0), (-5, 0, 0, 2, 1)\}$ dim. W = 3]

Example 2. *Let V_1 and V_2 be two sub-spaces of R^4 given by*

$$V_1 = \{(a, b, c, d) \mid b - 2c + d = 0\}$$

$$V_2 = \{(a, b, c, d) \mid a = d, b = 2c\}.$$

Find the basis and dimension of

(*i*) V_1 (*ii*) V_2 (*iii*) $V_1 \cap V_2$. (G.N.D.U. 1985)

Sol. We shall find the basis and dimension of the solution spaces of the following equations :

(*i*) $b - 2c + d = 0$ (*ii*) $a = d, b = 2c$ (*iii*) $b - 2c + d = 0, a = d, b = 2c$.

(*i*) V_1 is the solution space of $b - 2c + d = 0$, *i.e.*, $b = 2c - d$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ 2c-d \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} d$$

Here a, c and d are three independent variables and the solution space consists of three L.I. vectors, *viz.* (1, 0, 0, 0), (0, 2, 1, 0) and (0, – 1, 0, 1)

$\therefore$ Basis of $V_1 = \{(1, 0, 0, 0), (0, 2, 1, 0), (0, -1, 0, 1)\}$ and dim. $V_1 = 3$.

(*ii*) V_2 is the solution space of $a = b$, $b = 2c$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} d \\ 2c \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} d$$

Here c and d are two independent variables and the solution space consists of 2 L.I. vectors ; *viz.*

(0, 2, 1, 0) and (1, 0, 0, 1).

$\therefore$ Basis of $V_2 = \{(0, 2, 1, 0), (1, 0, 0, 1)\}$, and dim. $V_2 = 2$.

(*iii*) $V_1 \cap V_2$ is the solution space of the equations

$$a = d, b = 2c, b - 2c + d = 0$$

$$\therefore \quad d = 0, a = 0, b = 2c$$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 2c \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} c$$

Here c is the only independent variable and the solution space consists of only one L.I. vector, *viz.* (0, 2, 1, 0)

$\therefore$ Basis of $V_1 \cap V_2 = \{(0, 2, 1, 0)\}$ and dim. $V_1 \cap V_2 = 1$.

Example 3. *Let M and N be sub-spaces of R^4 defined as* :

$$M = \{(a, b, c, d) \mid b + c + d = 0\}$$

$$N = \{(a, b, c, d) \mid a + b = 0, c = 2d\}.$$

Find the dimension and basis of M, N and $M \cap N$. (G.N.D.U. 1988)

Sol. (*i*) We have $M = \{(a, b, c, d) \mid b + c + d = 0\}$

i.e., M is the solution space of equation

$$b + c + d = 0 \quad \text{or} \quad b = -c - d.$$

To find its basis :

Here $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ -c-d \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} d$$

Here there are four unknowns of which three *viz.* a, c, d are independent.

Hence basis of solution space $M = \{(1, 0, 0, 0), (0, -1, 1, 0), (0, -1, 0, 1)\}$ and dim. $M = 3$.

(*ii*) Here $N = \{(a, b, c, d) \mid a + b = 0, c = 2d\}$

i.e., N is the solution space of equations $a + b = 0,\ c = 2d$

or $a = -b, c = 2d.$

To find its basis :

Here $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -b \\ b \\ 2b \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} h + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} d$$

Here there are four unknowns of which b and d are independent.

Hence basis of solution space $N = \{(-1, 1, 0, 0), (0, 0, 2, 1)\}$ and dim. $N = 2$

(*iii*) Here $$\begin{aligned} M \cap N &= \{(a, b, c, d) \mid b + c + d = 0, a + b = 0, c = 2d\} \\ &= \{(a, b, c, d) \mid b + 2d + d = 0, a = -b, c = 2d\} \\ &= \{(a, b, c, d) \mid b = -3d, a = -b, c = 2d\} \\ &= \{(a, b, c, d) \mid a = 3d, b = -3d, c = 2d\}. \end{aligned}$$

To find its basis :

Here $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3d \\ -3d \\ 2d \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 1 \end{bmatrix} d$$

Here there are four unknowns of which only d is independent.

Here the basis of solution space $M \cap N = \{(3, -3, 2, 1)\}$ and dim. $(M \cap N) = 1$.

3

LINEAR TRANSFORMATIONS

1. Linear Transformation (L.T.)

Def. *Let V and W be two vector spaces over the same field F. Then a mapping T of V into W i.e.,*

$$T : V \to W$$

is called a linear transformation if the following properties are satisfied :

(I) Additive Property. $T(x+y) = T(x) + T(y) \ \forall\, x, y \in V$.

(II) Homogeneous Property.

$$T(\alpha x) = \alpha T(x) \qquad \forall\, x \in V \text{ and } \alpha \in F.$$

L.T. stands for the abbreviation of Linear Transformation, which is also known as **vector space homomorphism.**

The above-mentioned two properties are combined into a single property, namely **Linear Property** as :

$$T(\alpha x + y) = \alpha T(x) + T(y) \qquad \forall\, x, y \in V \text{ and } \alpha \in F$$

or $\quad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall\, x, y \in V \text{ and } \alpha, \beta \in F.$

2. Linear Operator (L.O.)

Def. *If in the above definition of linear transformation, the vector space W is the same as V, then the linear transformation*

$$T : V \to V$$

is called the linear operator.

Thus $\quad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall\, x, y \in V \text{ and } \alpha, \beta \in F.$

L.O. *stands for the abbreviation of Linear Operator, which is a mapping from one vector space into itself.*

3. Linear Functional

Let V (F) be a vector space and T be a mapping from V into F

i.e., $\qquad T : V \to F$

such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \ \forall\, x, y \in V$ *and* $\alpha, \beta \in F$, *then the mapping T is called linear functional on V.*

This is also known as **scalar valued function.**

THEOREMS

Theorem I. Zero Transformation (Operator). *If V (F) and W (F) are vector spaces, then a mapping T defined as*

$$T : V \to W,\ T(x) = 0 \ \forall\, x \in V$$

is a linear transformation (or operator).

Proof. First of all, $T : V \to W$ (or V) $\qquad$ [$\because$ **0** *is the common element of V and W*]

Now for $x, y \in V, \alpha, \beta \in F \ \Rightarrow \alpha x + \beta y \in V$ $\qquad$ [$\because$ *V is a V.S.*]

Thus by def. of T, we have

$$T(\alpha x + \beta y) = 0 = \alpha.0 + \beta.0 = \alpha . T(x) + \beta . T(y)$$

$\Rightarrow$ by def. of L.T., T is a L.T.

Hence T is a linear transformation (or operator) and it is called **zero transformation (or operator)**, denoted by **O**.

Remember : $\mathbf{O}(x) = \mathbf{0} \ \forall \ x \in V$.

Theorem II. Identity Operator. *If V (F) is a vector space, then the mapping T defined as*

$$T(x) = x \ \forall \ x \in V$$

is a linear operator. **(G.N.D.U. 1985 S)**

Proof. First of all, T : V $\to$ V is defined as $T(x) = x \ \forall \ x \in V$ ...(1)

Now, for $x, y \in V$ and $\alpha, \beta \in F \Rightarrow \alpha x + \beta y \in V$.

$$\therefore \quad T(\alpha x + \beta y) = \alpha x + \beta y \qquad [\because \ \textit{of (1)}]$$

$$= \alpha T(x) + \beta T(y)$$

$\Rightarrow$ T is a linear operator.

Hence T is a linear operator, called **Identity operator,** denoted by I.

Remember : $I(x) = x \ \forall \ x \in V$.

Theorem III. Negative of transformation (or operator). *If V(F) and W(F) are vector spaces and T is a linear transformation from V to W, then the mapping – T defined by*

$$(-T)x = -[T(x)] \ \forall \ x \in V$$

is a linear transformation.

Proof. First of all, – T : V $\to$ W is defined as

$$(-T)x = -[T(x)] \ \forall \ x \in V \qquad ...(1)$$

Now for $x, y \in V$ and $\alpha, \beta \in F \Rightarrow \alpha x + \beta y \in V$.

$$\therefore \quad (-T)(\alpha x + \beta y) = -[T(\alpha x + \beta y)] \qquad [\because \ \textit{of (1)}]$$

$$= -[\alpha T(x) + \beta T(y)] \qquad [\because \ T \text{ is L.T.}]$$

$$= -\alpha T(x) - \beta T(y)$$

$$= \alpha[-T(x)] + \beta[-T(y)]$$

$$= \alpha[(-T)x] + \beta[(-T)y]$$

$\Rightarrow$ – T is L.T.

Hence – T is a linear transformation corresponding to linear transformation T.

Theorem IV. Properties of linear transformations.

If the mapping $T : V \to W$ is a linear transformation from the vector space V (F) to the vector space W (F), then

(*i*) $T(\mathbf{0}) = \mathbf{0}$, *where left hand* $\mathbf{0} \in V$ *and right hand* $\mathbf{0} \in W$

(*ii*) $T(-x) = -T(x) \quad \forall \ x \in V$

(*iii*) $T(x - y) = T(x) - T(y) \ \forall \ x, y \in V$.

Proof. (*i*) $T(x) = x'$ for $x \in V, x' \in W$ ...(1)

Then $T(x) = T(x + \mathbf{0}) = T(x) + T(\mathbf{0})$ $[\because \ T \text{ is L.T.}]$

$\Rightarrow \quad x' = x' + T(0)$ [By (I)]

$\Rightarrow \quad x' + 0 = x' + T(0)$ [$\because$ $x' = x' + 0$]

$\Rightarrow \quad T(0) = 0.$ [Using Cancellation Law]

(ii) $T(-x) = T((-1)x)$ [$\because$ T is L.T.]

$= (-1)T(x)$ [By def.]

$= -T(x).$

(iii) $T(x-y) = T(x+(-y)) = T(x) + T(-y)$ [$\because$ T is L.T.]

$= T(x) - T(y).$ [Using (ii)]

Theorem V. Linear transformation for same dimensional vector spaces. *If V (F) and W (F) are n-dimensional vector spaces having their bases as*

$$B_1 = \{x_1, x_2, \ldots\ldots, x_n\} \text{ and } B_2 = \{y_1, y_2, \ldots\ldots, y_n\}$$

respectively, then there exists a unique linear transformation T from V (F) to W (F) such that

$$T(x_i) = y_i,\ i = 1, 2, \ldots\ldots, n. \quad \ldots(A)$$

Proof. Since B_1 is basis of V, so x, any vector in V, can be written as

$$x = \sum_{i=1}^{n} \alpha_i x_i \text{ for } \alpha_i\text{'s} \in F \quad \ldots(1)$$

(i) **To show the existence of T.**

We define a mapping

$$T(x) = T(\Sigma \alpha_i x_i) = \Sigma \alpha_i T(x_i) \quad \text{[By property of T]}$$

$$= \sum_{i=1}^{n} \alpha_i y_i \text{ for } \alpha_i\text{'s} \in F \text{ and } \forall x \in F \quad \ldots(2) \text{ (By (A))}$$

Now we show that T is a linear transformation.

Since $T : V \to W$ [$\because$ $\Sigma \alpha_i y_i \in W$ as B_2 is basis of W]

and for each $x_1, x_2 \in V$ and $\alpha, \beta \in F \Rightarrow \alpha x_1 + \beta x_2 \in V$,

where x_1 and x_2 can be written as

$$x_1 = \sum_{i=1}^{n} \beta_i x_i \text{ for } \beta_i\text{'s} \in F \text{ and } x_2 = \sum_{i=1}^{n} \gamma_i x_i \text{ for } \gamma_i\text{'s} \in F \quad \ldots(3)$$

$T(\alpha x_1 + \beta x_2) = T(\alpha \Sigma \beta_i x_i + \beta \Sigma \gamma_i x_i)$ [$\because$ of (3)]

$= T(\Sigma(\alpha\beta_i + \beta\gamma_i) x_i) = \Sigma(\alpha\beta_i + \beta\gamma_i) y_i$ [$\because$ of (2)]

$= \alpha \Sigma \beta_i y_i + \beta \Sigma \gamma_i y_i = \alpha T(x_1) + \beta T(x_2)$

$\Rightarrow$ T is linear.

Further particular case of (2) is given as

$x = x_i = 0 . x_1 + 0 . x_2 + \ldots + 1 . x_i + \ldots + 0 . x_n \in W$ for $i = 1, 2, \ldots, n$

i.e., $T(x_i) = 1 . y_i = y_i$ for $i = 1, 2, \ldots, n$.

Thus T is the linear transformation from $V \to W$ by the property (A) showing the existence of a linear transformation.

(*ii*) **Uniqueness of T.**

Let T' be another transformation such that

$$T'(x_i) = y_i, \text{ for } i = 1, 2, \ldots, n \qquad \ldots(4)$$

Then $\quad T'(x) = T'(\Sigma \alpha_i x_i)$ by (1) $\forall x \in V$

$$= \sum_{i=1}^{n} \alpha_i T'(x_i) \qquad [\textit{By linearity of } T']$$

$$= \sum_{i=1}^{n} \alpha_i y_i \qquad [\because \textit{ of } (4)]$$

$$= T(x). \qquad [\because \textit{ of } (2)]$$

As above equality is true for each $x \in V$ so

$$T = T'$$

showing the uniqueness of T.

Theorem VI. *Let $T : V \to W$ be a linear transformation and suppose $x_1, x_2, \ldots, x_n \in V$ have the property that their images $T(x_1), T(x_2), \ldots, T(x_n)$ are L.I. Show that the vectors $x_1, x_2, \ldots, x_n$ are L.I.*

(G.N.D.U. 1986 ; P.U. 1985 S)

Proof. Let there exist scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$$

Now $T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n) = T(0) = 0$

$\Rightarrow \quad \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots\ldots + \alpha_n T(x_n) = 0 \qquad [\because$ *T is L.T.*$]$

$\Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0 \qquad [\because T(x_1), T(x_2), \ldots\ldots, T(x_n)$ *are L.I.*$]$

Hence $x_1, x_2, \ldots\ldots, x_n$ are also L.I.

SOLVED EXAMPLES

Example 1. *Find out which of the following are linear transformations :*

(*i*) *$T : R^2 \to R$ defined by $T(x, y) = (xy)$*

(*ii*) *$T : R^2 \to R$ defined by $T(x, y) = (x - y)$*

(*iii*) *$T : R \to R^2$ defined by $T(x) = (2x, 3x)$* (P.U. 1996, 92 ; G.N.D.U. 1985 S)

(*iv*) *$T : R^3 \to R^3$ defined by $T(x, y, z) = (x + 1, y, z)$*

(*v*) *$T : R^3 \to R^2$ defined by $T(x, y, z) = (z, x + y)$*

(*vi*) *$f : R^2 \to R$ defined by $f(x, y) = |2x - 3y|$.* (G.N.D.U. 1985 S)

Sol. (*i*) We have $T : R^2 \to R$ defined by $T(x, y) = (x y)$.

Let $a = (x_1, y_1) \in V_2(R)$ and $b = (x_2, y_2) \in V_2(R)$ and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha (x_1, y_1) + \beta (x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$\therefore \quad T(\alpha a + \beta b) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (\alpha x_1 + \beta x_2)(\alpha y_1 + \beta y_2) \qquad \ldots(1)$

and $\alpha T(a)+\beta T(b)=\alpha T(x_1, y_1)+\beta T(x_2, y_2)=\alpha x_1 y_2+\beta x_2 y_2$

From (1) and (2), $T(\alpha a+\beta b) \neq \alpha T(a)+\beta T(b)$.

Hence T is not a linear transformation.

(*ii*) We have $T: R^2 \to R$ defined by $T(x, y)=(x-y)$.

Let $a=(x_1, y_1) \in V_2(R)$ and $b=(x_2, y_2) \in V_2(R)$

and α, β be any two real numbers.

$$\alpha a+\beta b=\alpha(x_1, y_1)+\beta(x_2, y_2)=(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2)$$

$$\therefore \quad T(\alpha a+\beta b)=T(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2)=(\alpha x_1+\beta x_2)-(\alpha y_1+\beta y_2)$$

$$=\alpha(x_1-y_1)+\beta(x_2-y_2)=\alpha T(x_1, y_1)+\beta T(x_2, y_2)$$

$$=\alpha T(a)+\beta T(b).$$

Hence T is a linear transformation.

(*iii*) We have $T: R \to R^2$ defined by $T(x)=(2x, 3x)$.

Let $a=(x_1) \in V_1(R)$ and $b=(x_2) \in V_1(R)$

and α, β be any two real numbers.

$$\alpha a+\beta b=\alpha(x_1)+\beta(x_2)=(\alpha x_1+\beta x_2)$$

$$\therefore \quad T(\alpha a+\beta b)=T(\alpha x_1+\beta x_2)=(2\alpha x_1+2\beta x_2, 3\alpha x_1+3\beta x_2) \quad \ldots(1)$$

And $\alpha T(a)+\beta T(b)=\alpha T(x_1)+\beta T(x_2)$

$$=\alpha(2x_1, 3x_1)+\beta(2x_2, 3x_2)=(2\alpha x_1, 3\alpha x_1)+(2\beta x_2+3\beta x_2)$$

$$=(2\alpha x_1+2\beta x_2, 3\alpha x_1+3\beta x_2) \quad \ldots(2)$$

From (1) and (2), $T(\alpha a+\beta b)=\alpha T(a)+\beta T(b)$

Hence T is a linear transformation.

(*iv*) We have $T: R^3 \to R^3$ defined by, $T(x, y, z)=(x+1, y, z)$

Let $a=(x_1, y_1, z_1) \in V_3(R)$

and $b=(x_2, y_2, z_2) \in V_3(R)$

and α, β be any two real numbers.

$$\alpha a+\beta b=\alpha(x_1, y_1, z_1)+\beta(x_2, y_2, z_2)=(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2)$$

$$\therefore \quad T(\alpha a+\beta b)=T(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2)$$

$$=(\alpha x_1+\beta x_2+1, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2) \quad \ldots(1)$$

And $\alpha T(a)+\beta T(b)=\alpha T(x_1, y_1, z_1)+\beta T(x_2, y_2, z_2)=\alpha(x_1+1, y_1, z_1)+\beta(x_2+1, y_2, z_2)$

$$=(\alpha x_1+\alpha, \alpha y_1, \alpha z_1)+(\beta x_2+\beta, \beta y_2, \beta z_2)$$

$$=(\alpha x_1+\beta x_2+\alpha+\beta, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2) \quad \ldots(2)$$

From (1) and (2), $T(\alpha a+\beta b) \neq \alpha T(a)+\beta T(b)$.

Hence T is not a linear transformation.

(*v*) We have $T: R^3 \to R^2$ defined by

$$T(x, y, z)=(z, x+y)$$

Let $a=(x_1, y_1, z_1) \in V_3(R)$ and $b=(x_2, y_2, z_2) \in V_3(R)$ and α, β be any two real numbers.

$$\alpha a+\beta b=\alpha(x_1, y_1, z_1)+\beta(x_2, y_2, z_2)=(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2)$$

$$\therefore \ T(\alpha a+\beta b)=(\alpha z_1+\beta z_2, \alpha x_1+\beta x_2+\alpha y_1+\beta y_2)$$

$$=(\alpha z_1+\beta z_2, \alpha(x_1+y_1)+\beta(x_2+y_2)) \quad \ldots(1)$$

and $\alpha T(a)+\beta T(b) = \alpha\, T(x_1, y_1, z_1) + \beta\, T(x_2, y_2, z_2) = \alpha\,(z_1, x_1+y_1) + \beta\,(z_2, x_2 + y_2)$

$$= (\alpha z_1, \alpha(x_1+y_1)) + (\beta z_2, \beta(x_2+y_2))$$

$$= (\alpha z_1 + \beta z_2, \alpha(x_1+y_1) + \beta(x_2+y_2)) \quad ...(2)$$

From (1) and (2), $T(\alpha a+\beta b) = \alpha T(a) + \beta T(b)$.

Hence T is a linear transformation.

(*vi*) We have

$f : R^2 \to R$ defined by $f(x, y) = |2x - 3y|$.

Let $a = (x_1, y_1) \in V_2(R)$ and $b = (x_2, y_2) \in V_2(R)$

and α, β are any two real numbers.

$$\alpha a + \beta b = \alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$\therefore \quad f(\alpha a+\beta b) = f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$$= |2(\alpha x_1 + \beta x_2) - 3(\alpha y_1 + \beta y_2)|$$

$$= |\alpha(2x_1 - 3y_1) + \beta(2x_2 - 3y_2)| \quad ...(1)$$

And $\alpha f(a) + \beta f(b) = \alpha f(x_1, y_1) + \beta f(\alpha_2, y_2)$

$$= \alpha\,|2x_1 - 3y_1| + \beta\,|2x_2 - 3y_2|$$

$$= |\alpha(2x_1 - 3y_1)| + |\beta(2x_2 - 3y_2)| \quad ...(2)$$

From (1) and (2), $f(\alpha a + \beta b) \neq \alpha f(a) + \beta f(b)$

Hence $f : R^2 \to R$ is not a linear transformation.

Example 2. *Show that the following mappings T are linear transformations :*

(*i*) $T : R^2 \to R^2$ *defined by* $T(x, y) = (x + y, x)$

(*ii*) $T : R^2 \to R^3$ *defined by* $T(x, y) = (x + y, x - y, y)$

(*iii*) $T : R^3 \to R^2$ *defined by* $T(x, y, z) = (x - y, x - z)$

(*iv*) $T : R^3 \to R^2$ *defined by* $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$ (G.N.D.U. 1996)

(*v*) $T : R^3 \to R$ *defined by* $T(x, y, z) = (2x - 3y + 4z)$.

Sol. (*i*) We have $T : R^2 \to R^2$ defined by

$$T(x, y) = (x + y, x)$$

Let $a = (x_1, y_1) \in V_2(R)$, $b = (x_2, y_2) \in V_2(R)$

and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$\therefore \quad T(\alpha a + \beta b) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2) \quad ...(1)$$

And $\alpha T(a) + \beta T(b) = \alpha T(x_1, y_1) + \beta T(x_2, y_2) = \alpha(x_1 + y_1, x_1) + \beta(x_2 + y_2, x_2)$

$$= (\alpha x_1 + \alpha y_1, \alpha x_1) + (\beta x_2 + \beta y_2, \beta x_2)$$

$$= (\alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2, \alpha x_1 + \beta x_2)$$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2) \quad ...(2)$$

From (1) and (2), $T(\alpha a + \beta b) = \alpha\, T(a) + \beta\, T(b)$.

Hence T is a linear transformation.

(*ii*) We have $T : R^2 \to R^3$ defined by

$$T(x, y) = (x + y, x - y, y)$$

Let $a = (x_1, y_1) \in V_2(R)$, $b = (x_2, y_2) \in V_2(R)$

and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha (x_1, y_1) + \beta (x_2, y_2)$$
$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$\therefore \quad T(\alpha a + \beta b) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$
$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2, \alpha y_1 + \beta y_2) \quad \ldots(1)$$

And $\alpha T(a) + \beta T(b) = \alpha\, T(x_1, y_1) + \beta\, T(x_2, y_2)$

$$= \alpha (x_1 + y_1, x_1 - y_1, y_1) + \beta (x_2 + y_2, x_2 - y_2, y_2)$$
$$= (\alpha x_1 + \alpha y_1, \alpha x_1 - \alpha y_1, \alpha y_1) + (\beta x_2 + \beta y_2, \beta x_2 - \beta y_2, \beta y_2)$$
$$= (\alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2, \alpha x_1 - \alpha y_1 + \beta x_2 - \beta y_2, \alpha y_1 + \beta y_2)$$
$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2, \alpha y_1 + \beta y_2) \quad \ldots(2)$$

From (1) and (2), $T(\alpha a + \beta b) = \alpha T(a) + \beta T(b)$.

Hence T is a linear transformation.

(*iii*) We have $T : R^3 \to R^2$ defined by

$$T(x, y, z) = (x - y, x - z)$$

Let $a = (x_1, y_1, z_1) \in V_3(R)$,

$b = (x_2, y_2, z_2) \in V_3(R)$

and α, β be any two real numbers.

$$\alpha x + \beta b = \alpha (x_1, y_1, z_1) + \beta (x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$\therefore \quad T(\alpha a + \beta b) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$
$$= (\alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2, \alpha x_1 + \beta x_2 - \alpha z_1 - \beta z_2) \quad \ldots(1)$$

And $\alpha T(a) + \beta T(b) = \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$

$$= \alpha (x_1 - y_1, x_1 - z_1) + \beta (x_2 - y_2, x_2 - z_2)$$
$$= (\alpha x_1 - \alpha y_1, \alpha x_1 - \alpha z_1) + (\beta x_2 - \beta y_2, \beta x_2 - \beta z_2)$$
$$= (\alpha x_2 - \alpha y_1 + \beta x_2 - \beta y_2, \alpha x_1 - \alpha z_1 + \beta x_2 - \beta z_2)$$
$$= (\alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2, \alpha x_1 + \beta x_2 - \alpha z_1 - \beta z_2) \quad \ldots(2)$$

From (1) and (2), $T(\alpha a + \beta b) = \alpha T(a) + \beta\, T(b)$.

Hence T is a linear transformation.

(*iv*) We have $T : R^3 \to R^2$ defined by

$$T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$$

Let $a = (x_1, y_1, z_1) \in V_3(R)$, $b = (x_2, y_2, z_2) \in V_3(R)$

and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha (x_1, y_1, z_1) + \beta (x_2, y_2, z_2)$$
$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$\therefore \quad T(\alpha a + \beta b) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$
$$= \big(3(\alpha x_1 + \beta x_2) - 2(\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2), \alpha x_1 + \beta x_1 - 3(\alpha y_1 + \beta y_2) - 2(\alpha z_1 + \beta z_2)\big)$$

$$= \left(\alpha\,(3x_1 - 2y_1 + z_1) + \beta\,(3x_2 - 2y_2 + z_2),\ \alpha\,(x_1 - 3y_1 - 2z_1) + \beta\,(x_2 - 3y_2 - 2z_2)\right)$$
$$= \left(\alpha\,(3x_1 - 2y_1 + z_1),\ \alpha\,(x_1 - 3y_1 - 2z_1)\right) + \left(\beta\,(3x_2 - 2y_2 + z_2) + \beta\,(x_2 - 3y_2 - 2z_2)\right)$$
$$= \alpha\,(3x_1 - 2y_1 + z_1, x_1 - 3y_1 - 2z_1) + \beta\,(3x_2 - 2y_2 + z_2, x_2 - 3y_2 - 2z_2)$$
$$= \alpha\,T\,(x_1, y_1, z_1) + \beta\,T\,(x_2, y_2, z_2) = \alpha T(a) + \beta T(b).$$

Hence T is a linear transformation.

(*v*) We have $T : R^3 \to R$ defined by

$$T\,(x, y, z) = (2x - 3y + 4z)$$

Let $a = (x_1, y_1, z_1) \in V_3\,(R)$, $b = (x_2, y_2, z_2) \in V_3\,(R)$

and α, β be any two real numbers,

$$\alpha a + \beta b = \alpha\,(x_1, y_1, z_1) + \beta\,(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$\therefore \quad T\,(\alpha a + \beta b) = T\,(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$
$$= \left(2\,(\alpha x_1 + \beta x_2) - 3\,(\alpha y_1 + \beta y_2) + 4\,(\alpha z_1 + \beta z_2)\right)$$
$$= \alpha\,(2x_1 - 3y_1 + 4z_1) + \beta\,(2x_2 - 3y_2 + 4z_2)$$
$$= \alpha T\,(x_1, y_1, z_1) + \beta T\,(x_2, y_2, z_2)$$
$$= \alpha T\,(a) + \beta T\,(b)$$

Hence T is a linear transformation.

Example 3. *Show that the following mappings T are not linear transformations* :

(*i*) $T : R^3 \to R^2$ *defined by* $(x, y, z) = (|\,x\,|, 0)$

(*ii*) $T : R^2 \to R^3$ *defined by* $T\,(x, y) = (x + 1, 2y, x + y)$.

Sol. (*i*) We have $T : R^3 \to R^2$ defined by

$$T\,(x, y, z) = (|\,x\,|, 0)$$

Let $a = (x_1, y_1, z_1) \in V_3\,(R)$, $b = (x_2, y_2, z_2) \in V_3\,(R)$

and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha\,(x_1, y_1, z_1) + \beta\,(x_2, y_2, z_2) = (\alpha\,x_1 + \beta\,x_2, \alpha\,y_1 + \beta\,y_2, \alpha\,z_1 + \beta\,z_2)$$

$$\therefore \quad T\,(\alpha a + \beta b) = T\,(\alpha\,x_1 + \beta\,x_1, \alpha\,y_1 + \beta\,y_2, \alpha\,z_1 + \beta\,z_2)$$
$$= (\,|\,\alpha\,x_1 + \beta\,x_2\,|, 0) \qquad \ldots(1)$$

And $\alpha T(a) + \beta T(b) = \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2) = \alpha\,(\,|\,x_1\,|, 0) + \beta\,(\,|\,x_2\,|, 0)$

$$= (\alpha\,|\,x_1\,| + \beta\,|\,x_2\,|, 0) \qquad \ldots(2)$$

From (1) and (2), $T\,(\alpha a + \beta b) \neq \alpha\,T\,(a) + \beta\,T\,(b)$.

Hence T is not a linear transformation.

(*ii*) We have $T : R^2 \to R^3$ defined by $T\,(x, y) = (x + 1, 2y, x + y)$

Let $a = (x_1, y_1) \in V_2\,(R)$, $b = (x_2, y_2) \in V_2\,(R)$

and α, β be any two real numbers.

$$\alpha a + \beta b = \alpha\,(x_1, y_1) + \beta\,(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$\therefore\ T\,(\alpha a + \beta b) = (\alpha x_1 + \beta x_2 + 1, 2\alpha y_1 + 2\beta y_2, \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2) \qquad \ldots(1)$$

and $\alpha\,T\,(a) + \beta\,T\,(b) = \alpha T(x_1, y_1) + \beta T(x_2, y_2)$

$$= \alpha\,(x_1 + 1, 2y_1, x_1 + y_1) + \beta\,(x_2 + 1, 2y_2, x_2 + y_2)$$
$$= (\alpha x_1 + \beta x_2 + \alpha + \beta, 2\,(y_1 + y_2), \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2) \qquad \ldots(2)$$

From (1) and (2), $T\,(\alpha a + \beta b) \neq \alpha T(a) + \beta(b)$.

Hence T is not a linear transformation.

Example 4. *Let $V(F)$ be a vector space of all $m \times n$ matrices over a field F and let P and Q be two fixed matrices of order $m \times m$ and $n \times n$ respectively over the same field F.*

$T : V \to V$ *defined by* $T(A) = PAQ \ \forall A \in V$.

Show that T is a linear transformation.

Sol. Since P is $m \times m$ and A is $m \times n$,

$\therefore$ PA is $m \times n$.

Also Q is $n \times n$.

$\therefore$ PAQ is $m \times n$ and thus PAQ $\in$ V (F).

Let A, B $\in$ V and $\alpha, \beta \in$ F

$\therefore$ αA and βB are also matrices of order $m \times n$

$\Rightarrow$ $\alpha A + \beta B \in V$ [$\because$ *A and B are $m \times n$*]

$$\begin{aligned} \therefore \quad T(\alpha A + \beta B) &= P(\alpha A + \beta B) Q \\ &= (P(\alpha A) + P(\beta B)) Q = (\alpha PA + \beta PB) Q \\ &= (\alpha PA) Q + (\beta PB) Q = \alpha (PAQ) + \beta (PBQ) \\ &= \alpha T(A) + \beta T(B). \end{aligned}$$

Hence T is a linear transformation.

Example 5. *Let $V(F)$ be a vector space of $n \times n$ matrices. Let M be a fixed $n \times n$ matrix. Then the mapping defined by*

(*i*) $T(A) = AM + MA \ \forall A \in V$.

(*ii*) $T(A) = AM - MA \ \forall A \in V$

are linear transformations.

Sol. Since each element of V is a $n \times n$ matrix, therefore, their product will also be a $n \times n$ matrix

i.e., A $\in$ V, M $\in$ V $\Rightarrow$ AM, MA $\in$ V

$\Rightarrow$ (AM + MA) $\in$ V and (AM − MA) $\in$ V [$\because$ *V is a vector space*]

Thus T : V $\to$ V.

Also for $A_1, A_2 \in V$

and $\alpha, \beta \in F$

$\Rightarrow$ $\alpha A_1 + \beta A_2 \in V$.

$$\begin{aligned} (i) \quad T(\alpha A_1 + \beta A_2) &= (\alpha A_1 + \beta A_2) M + M(\alpha A_1 + \beta A_2) \\ &= \alpha (A_1 M) + \beta (A_2 M) + \alpha (MA_1) + \beta (MA_2) \\ &= \alpha (A_1 M + MA_1) + \beta (A_2 M + MA_2) \\ &= \alpha T(A_1) + \beta T(A_2) \end{aligned}$$

Hence T is a linear transformation.

$$\begin{aligned} (ii) \quad T(\alpha A_1 + \beta A_2) &= (\alpha A_1 + \beta A_2) M - M(\alpha A_1 + \beta A_2) \\ &= \alpha (A_1 M) + \beta (A_2 M) - \alpha (MA_1) - \beta (MA_2) \\ &= \alpha (A_1 M - MA_1) + \beta (A_2 M - MA_2) \\ &= \alpha T(A_1) + \beta T(A_2) \end{aligned}$$

Hence T is a linear transformation.

Example 6. *Let F be a field and let V be the space of polynomial functions f from F into F, defined by*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n$$

Let $(Df)(x) = a_1 + 2a_2 x + \ldots\ldots + n a_n x^{n-1}$

Prove that D is a linear transformation from V into V.

Sol. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$

and $g(x) = b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_m x^m$

be two polynomials $\in$ V.

Let $\alpha, \beta \in$ F.

$$\therefore \quad \alpha f(x) + \beta g(x) = \alpha(a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n) + \beta(b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_m x^m)$$
$$= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1) x + (\alpha a_2 + \beta b_2) x^2 + \ldots\ldots$$

$$\therefore \quad D(\alpha f(x) + \beta g(x))$$
$$= (\alpha a_1 + \beta b_1) + 2(\alpha a_2 + \beta b_2) x + \ldots\ldots$$
$$= (\alpha a_1 + 2\alpha a_2 x + \ldots\ldots) + (\beta b_1 + 2\beta b_2 x + \ldots\ldots)$$
$$= \alpha(a_1 + 2a_2 x + \ldots\ldots) + \beta(b_1 + 2b_2 x + \ldots\ldots)$$
$$= \alpha(Df)x + \beta(Dg)x.$$

Hence D is a linear transformation.

Example 7. *Let $V(R)$ be a vector space of integrable functions on R. Define $T : V \to R : T(f) = \int_a^b f(x)\,dx$, $f \in V, a, b \in R$. Prove that T is a functional.*

Sol. Since T maps elements of V to the field F,

$\therefore$ T is functional.

To prove. T is linear.

Let $f_1, f_2 \in$ V and $c_2, c_2 \in$ R so that $c_1 f_1 + c_2 f_2 \in$ V.

$$\text{Now } T(c_1 f_1 + c_2 f_2) = \int_a^b (c_1 f_1(x) + c_2 f_2(x))\,dx$$
$$= c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx$$
$$= c_1 T(f_1) + c_2 T(f_2).$$

Hence T is linear.

Example 8. *Let R be the field of real numbers and V be the space of all functions from R into R which are continuous.*

Now T is defined as $(Tf)x = \int_0^x f(t)\,dt.$

Prove that T is a L.T. from V into V.

Sol. Let $f(x)$ and $g(x)$ be two functions in V and $\alpha, \beta \in$ R.

Then $\alpha f(x) + \beta g(x) \in$ V.

Since V is a vector space,

$$\therefore \quad (\alpha f + \beta g)x = \alpha f(x) + \beta g(x)$$

$$\Rightarrow \quad (T(\alpha f + \beta g))x = \int_0^x (\alpha f + \beta g)(t)\,dt = \int_0^x (\alpha f(t)\,dt + \beta g(t))\,dt$$

$$= \int_0^x \alpha f(t)\,dt + \int_0^x \beta g(t)\,dt = \alpha \int_0^x f(t)\,dt + \beta \int_0^x g(t)\,dt$$

$$= \alpha\,(\mathrm{T}f)\,x + \beta\,(\mathrm{T}\,g)\,x.$$

Hence T is a linear transformation.

Example 9. *Describe explicitly the linear transformation $T : R^2 \to R^2$ such that $T(e_1) = (a, b)$; $T(e_2) = (c, d)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are unit vectors.*

Sol. Let $\mathrm{T}(x_1, x_2) \in \mathrm{R}^2$.

We are to find $\mathrm{T}(x_1, x_2)$ subject to the conditions

$\mathrm{T}(e_1) = (a, b)$ and $\mathrm{T}(e_2) = (c, d)$.

We know that $\{e_1, e_2\}$ is a basis set for R^2,

$\therefore$ any vector $(x_1, x_2) \in \mathrm{R}^2$ can be expressed as a linear combination of the elements of the basis set.

Now $(x_1, x_2) = x_1(1, 0) + x_2(0, 1) = x_1 e_1 + x_2 e_2$

$\therefore$ $\mathrm{T}(x_1, x_2) = \mathrm{T}(x_1 e_1 + x_2 e_2) = x_1\,\mathrm{T}(e_1) + x_2\,\mathrm{T}(e_2) = x_1(a, b) + x_2(c, d)$

$$= (a x_1, b x_1) + (c x_2, d x_2) = (a x_1 + c x_2, b x_1 + d x_2),$$

which is the required L.T.

Example 10. *Describe explicitly the linear transformation $T : R^2 \to R^2$ such that $T(2, 3) = (4, 5)$ and $T(1, 0) = (0, 0)$.*

Sol. First of all, we shall establish the given vectors of domain of T form a basis for the domain of T, *i.e.*, R^2.

To prove : *(2, 3) and (1, 0) are L.I.*

Let $a(2, 3) + b(1, 0) = \mathbf{0} = (0, 0)$

$\Rightarrow$ $(2a + b, 3a + 0) = (0, 0)$ $\Rightarrow$ $(2a + b, 3a) = (0, 0)$

$\Rightarrow$ $2a + b = 0, 3a = 0$ $\Rightarrow$ $a = 0, b = 0$.

Thus (2, 3) and (1, 0) are L.I.

To prove : *(2, 3) and (1, 0) span R^2.*

Let $(x_1, x_2) \in \mathrm{R}^2$.

Let $(x_1, x_2) = a(2, 3) + b(1, 0) = (2a + b, 3a)$

$\Rightarrow$ $2a + b = x_1,\quad 3a = x_2$

$$\Rightarrow \quad a = \frac{x_2}{3},\ b = \frac{3x_1 - 2x_2}{3}$$

$$\Rightarrow \quad (x_1, x_2) = \frac{x_2}{3}(2, 3) + \frac{3x_1 - 2x_2}{3}(1, 0)$$

Thus (2, 3) and (1, 0) span R^2.

$$\text{Hence,}\quad \mathrm{T}(x_1, x_2) = \mathrm{T}\left[\frac{x_2}{2}(2, 3) + \frac{3x_1 - 2x_2}{3}(1, 0)\right] = \frac{x_2}{3}\mathrm{T}(2, 3) + \frac{3x_1 - 2x_2}{3}\mathrm{T}(1, 0)$$

$$= \frac{x_2}{3}(4, 5) + \frac{3x_1 - 2x_2}{3}(0, 0) = \left(\frac{4x_2}{3}, \frac{5x_2}{3}\right),$$

which is the required L.T.

Example 11. *A linear transformation T of R^3 itself is defined by $T(e_1) = e_1 + e_2 + e_3$, $T(e_2) = e_2 + e_3$ and $T(e_3) = e_2 - e_3$, where e_1, e_2, e_3 are unit vectors of R^3.*

(*i*) *Determine the transformation of (2, – 1, 3).*

(*ii*) *Describe explicitly the linear transformation T.*

Sol. Since e_1, e_2, e_3 are unit vectors of R^3,

$\therefore$ $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

We have $T(e_1) = e_1 + e_2 + e_3 \Rightarrow T(e_1) = (1, 1, 1)$

$T(e_2) = e_2 + e_3 \Rightarrow T(e_3) = (0, 1, 1)$

$T(e_3) = e_2 - e_3 \Rightarrow T(e_3) = (0, 1, -1)$.

Since e_1, e_2, e_3 form a basis of R^3,

$\therefore$ every vector of R^3 can be uniquely expressed as a linear combination of e_1, e_2, e_3.

(*i*) Now $(2, -1, 3) = 2(1, 0, 0) + (-1)(0, 1, 0) + 3(0, 0, 1)$

$= 2e_1 + (-1)e_2 + 3e_3$

$\therefore$ $T(2, -1, 3) = T(2e_1 + (-1)e_2 + 3e_3) = 2T(e_1) + (-1)T(e_2) + 3T(e_3)$

$= 2(1, 1, 1) + (-1)(0, 1, 1) + 3(0, 1, -1)$

$= (2, 4, -2)$.

(*ii*) $(x, y, z) \in R^3$.

Now $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = xe_1 + ye_2 + ze_3$

$\therefore$ $T(x, y, z) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3)$

$= x(1, 1, 1) + y(0, 1, 1) + z(0, 1, -1)$

$= (x, x + y + z, x + y - z)$,

which describes completely the given linear transformation.

Example 12. *Find (x_1, x_2), where $T : R^2 \to R^3$ is defined by*

(*i*) *$T(1, 2) = (3, -1, 5)$, $T(0, 1) = (2, 1, -1)$*

(*ii*) *$T(2, -5) = (-1, 2, 3)$, $T(3, 4) = (0, 1, 5)$.* **(P.U. 1986)**

Sol. (*i*) **To prove :** *(1, 2), (0, 1) are L.I.*

Let $a(1, 2) + b(0, 1) = 0 = (0, 0)$

$\Rightarrow$ $(a + 0b, 2a + b) = (0, 0)$

$\Rightarrow$ $a = 0, \quad 2a + b = 0$

$\Rightarrow$ $a = 0, \quad b = 0$.

Thus (1, 2) and (0, 1) are L.I.

To prove : *(1, 2) and (0, 1) span R^3.*

Let $(x_1, x_2) \in R^2$.

Let $(x_1, x_2) = a(1, 2) + b(0, 1) = (a, 2a + b)$

$\Rightarrow$ $a = x_1, 2a + b = x_2$

$\Rightarrow$ $a = x_1, \quad b = x_2 - 2x_1$

$\therefore$ $(x_1, x_2) = x_1(1, 2) + (x_2 - 2x_1)(0, 1)$

Thus (1, 2) and (0, 1) span R^3.

Hence $T(x_1, x_2) = T[x_1, (1, 2) + (x_2 - 2x_1)(0, 1)] = x_1 T(1, 2) + (x_2 - 2x_1) T(0, 1)$

$= x_1(3, -1, 5) + (x_2 - 2x_1)(2, 1, -1)$

$= (3x_1 + 2x_2 - 4x_1, -x_1 + x_2 - 2x_1, 5x_1 - x_2 + 2x_1)$

$= (-x_1 + 2x_2, -3x_1 + x_2, 7x_1 - x_2)$.

(*ii*) **To prove :** *(2, – 5), (3, 4) are L. I.*

Let $a(2,-5)+b(3,4)=0=(0,0)$

$\Rightarrow \quad (2a+3b,-5a+4b)=(0,0)$

$\Rightarrow \quad 2a+3b=0,\ -5a+4b=0$

$\Rightarrow \quad a=0, b=0.$

Thus (2, – 5) and (3, 4) are L.I.

To prove : (2, – 5) and (3, 4) span R^3.

Let $(x_1, x_2) \in R^2$.

Let $(x_1,x_2)=a(2,-5)+b(3,4)=(2a+3b,\ -5a+4b)$

$\Rightarrow \quad 2a+3b=x_1,\ -5a+4b=x_2$

Solving, $a=\dfrac{4x_1-3x_2}{23}$ and $b=\dfrac{5x_1+2x_2}{23}$

$$\therefore \quad (x_1,x_2)=\frac{4x_1-3x_2}{23}(2,-5)+\frac{5x_1+2x_2}{23}(3,4)$$

Thus (2, – 5) and (3, 4) span R^3.

$$\begin{aligned}\text{Hence } T(x_1,x_2) &= T\left[\frac{4x_1-3x_2}{23}(2,-5)+\frac{5x_1+2x_2}{23}(3,4)\right]\\ &= \frac{4x_1-3x_2}{23}T(2,-5)+\frac{5x_1+2x_2}{23}T(3,4)\\ &= \frac{4x_1-3x_2}{23}(-1,2,3)+\frac{5x_1+2x_2}{23}(0,1,5)\\ &= \left(\frac{-4x_1+3x_2}{23},\frac{8x_1-6x_2}{23}+\frac{5x_1+2x_2}{23},\frac{12x_1-9x_2}{23}+\frac{25x_1+10x_2}{23}\right)\\ &= \left(\frac{-4x_1+3x_2}{23},\frac{13x_1-4x_2}{23},\frac{37x_1+x_2}{23}\right).\end{aligned}$$

Example 13. *Find $T(x_1, x_2, x_3)$, where $R^3 \to R$ is defined by*

$T(1, 1, 1)=3,\ T(0, 1, -2)=1,\ T(0, 0, 1)=-2.$

Sol. To prove : *(1, 1, 1), (0, 1, – 2) and (0, 0, 1) are L.I.*

Let $a(1,1,1)+b(0,1,-2)+c(0,0,1)=0=(0,0,0)$

$\Rightarrow \quad (a, a+b, a-2b+c)=(0,0,0)$

$\Rightarrow \quad a=0, a+b=0, a-2b+c=0$

$\Rightarrow \quad a=0, b=0, c=0.$

Thus (1, 1, 1), (0, 1, – 2) and (0, 0, 1) are L.I.

To prove : *(1, 1, 1), (0, 1, – 2) and (0, 0, 1) span R^3.*

Let $(x_1, x_2, x_3) \in R^3$.

$$\begin{aligned}\text{Let } (x_1,x_2,x_3) &= a(1,1,1)+b(0,1,-2)+c(0,0,1)\\ &= (a, a+b, a-2b+c)\end{aligned}$$

$\Rightarrow a=x_1, a+b=x_2, a-2b+c=x_3$

$\Rightarrow \quad a=x_1, b=x_2-x_1, c=x_3-x_1+2(x_2-x_1)=x_3+2x_2-3x_1.$

Thus (1, 1, 1), (0, 1, – 2) and (0, 0, 1) span R^3.

Hence $T(x_1, x_2, x_3) = T[x_1(1, 1, 1) + (x_2 - x_1)(0, 1, -2) + (x_3 + 2x_2 - 3x_1)(0, 1, 1)]$
$= x_1 T(1, 1, 1) + (x_2 - x_1) T(0, 1, -2) + (x_3 + 2x_2 - 3x_1) T(0, 0, 1)$
$= x_1(3) + (x_2 - x_1)(1) + (x_3 + 2x_2 - 3x_1)(-2)$
$= 3x_1 + x_2 - x_1 - 2x_3 - 4x_2 + 6x_1$
$= 8x_1 - 3x_2 - 2x_3.$

Example 14. *Find a linear transformation $T : R^2 \to R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$. Prove that T maps the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1) into a parallelogram.*

(G.N.D.U. 1998, 88 S)

Sol. As in Ex. 13,

$$T(x_1, x_2) = T[x_1(1, 0) + x_2(0, 1)] = x_1 T(1, 0) + x_2 T(0, 1)$$
$$= x_1(1, 1) + x_2(-1, 2)$$
$$= (x_1 - x_2, x_1 + 2x_2) \quad \text{...(1)}$$

Let the given vertices of the square be A, B, C, D respectively and let A′, B′, C′, D′ be their T images.

$\therefore$ $A' = T(A) = T(0, 0) = (0, 0)$ [*Putting $x_1 = 0, x_2 = 0$ in (1)*]

$B' = T(B) = T(1, 0) = (1, 1)$ [*Putting $x_1 = 1, x_2 = 0$ in (1)*]

$C' = T(C) = T(1, 1) = (0, 3)$ [*Putting $x_1 = 1, x_2 = 1$ in (1)*]

$D' = T(D) = T(0, 1) = (-1, 2)$ [*Putting $x_1 = 0, x_2 = 1$ in (1)*]

$\therefore$ $|A'B'| = \sqrt{(1-0)^2 + (1+0)^2} = \sqrt{1+1} = \sqrt{2}$

and $|C'D'| = \sqrt{(-1-0)^2 + (2-3)^2} = \sqrt{1+1} = \sqrt{2}$

Thus $|A'B'| = |C'D'| = \sqrt{2}$

Also $|A'D'| = |B'C'| = \sqrt{5}$. [*Verify !*]

Also slope of A′B′ = slope of C′D′ = 1.

Hence T maps into a parallelogram.

Example 15. *If V (F) and W (F) be two vector spaces and T_1, T_2 are linear transformations from V into W, prove that the mapping T defined by*

$$T(\alpha) = c\,T_1(\alpha) + T_2(\alpha), \alpha \in V, c \in F$$

is a linear transformation from V into W.

Sol. We have $T_1 : V \to W$, $T_2 : V \to W$ are linear transformations.

$\therefore$ $T_1(\alpha) \in W$, $T_2(\alpha) \in W \ \forall \alpha \in V$.

$\therefore$ $T(\alpha) = cT_1(\alpha) + T_2(\alpha)$ also belongs to W $\forall \alpha \in V$ because V is a vector space.

Hence T is a mapping from V into W.

To prove : *T is linear.*

Let $\alpha, \beta \in V$ and $k \in F$,

then $k\alpha + \beta \in V$

$\therefore$ $T(k\alpha + \beta) = cT_1(k\alpha + \beta) + T_2(k\alpha + \beta)$ [*Def. of T*]

$= c[kT_1(\alpha) + T_1(\beta)] + kT_2(\alpha) + T_2(\beta)$

[$\because$ *T_1 and T_2 are linear transformations*]

$= ckT_1(\alpha) + cT_1(\beta) + kT_2(\alpha) + T_2(\beta)$ [*Using Distributive Law*]

$= k\,[c\mathrm{T}_1(\alpha) + c\mathrm{T}_1(\beta)] + k\mathrm{T}_2(\alpha) + \mathrm{T}_2(\beta)$ [$\because$ $ck = kc$]

$= k\,[c\mathrm{T}_1(\alpha) + \mathrm{T}_2(\alpha)] + c\mathrm{T}_1(\beta) + \mathrm{T}_2(\beta)$

$= k\,\mathrm{T}(\alpha) + \mathrm{T}(\beta)$ [*Def. of T*]

Thus $\mathrm{T}(k\alpha + \beta) = k\,\mathrm{T}(\alpha) + \mathrm{T}(\beta)$.

Hence T is a linear transformation.

4. Range

Def. *Let U (F) and V (F) be two vector spaces and let*

$$T : U \to V$$

be a linear transformation, then the image set of U under V is called the range of T.

This is usually denoted by R (T).

Symbolically. Range (T) = $\{\mathrm{T}(x) : x \in \mathrm{U}\}$.

Theorem. *If T is a linear transformation from one vector space U (F) to another vector space V (F), then the range set T (U) of T is a sub-sapce of V (F).* (P.U. 1986 S, 85)

Proof. $\mathrm{T}(\mathrm{U}) = \{\mathrm{T}(x) : x \in \mathrm{U}\}$.

Since $\mathrm{T} : \mathrm{U} \to \mathrm{V}$ and $\forall\, x \in \mathrm{U} \;\Rightarrow\; \mathrm{T}(x) \in \mathrm{V}$,

$\therefore\quad \mathrm{T}(\mathrm{U}) \subset \mathrm{V}$.

Let $x, y \in \mathrm{T}(\mathrm{U})$, then

$x \in \mathrm{T}(\mathrm{U}) \;\Rightarrow\; \exists\, x_1 \in \mathrm{U}$ s.t. $\mathrm{T}(x_1) = x$,

$y \in \mathrm{T}(\mathrm{U}) \;\Rightarrow\; \exists\, y_1 \in \mathrm{U}$ s.t. $\mathrm{T}(y_1) = y$.

Let $\alpha, \beta \in \mathrm{F}$. $\therefore\; \alpha x_1 + \beta y_1 \in \mathrm{U}$ for $x_1, y_1 \in \mathrm{U}$ [$\because$ *U (F) is a vector space*]

$\therefore\quad \mathrm{T}(\alpha x + \beta y) = \alpha\,\mathrm{T}(x_1) + \beta\,\mathrm{T}(y_1)$

$= \mathrm{T}(\alpha x_1 + \beta y_1)$ [*By linearity of T*]

$\Rightarrow\quad \alpha x + \beta y \in \mathrm{T}(\mathrm{U})$ for $x, y \in \mathrm{T}(\mathrm{U})$ and $\alpha, \beta \in \mathrm{F}$.

Hence T (U) is a sub-space of W (F).

Range : Range of T is a vector space so it is called **range space.**

5. Null Space/Kernal

Def. *Let U (F) and V (F) be two vector spaces and T : U → V is a linear transformation. Then the set of all those vectors in U whose image under T is zero, is called the null space or kernel of T.*

This is usually denoted by N (T).

Symbolically. $\mathrm{N}(\mathrm{T}) = \{x : x \in \mathrm{U}, \mathrm{T}(x) = \mathbf{0} \in \mathrm{V}\}$.

Theorem. *Let U (F) and V (F) be two vector spaces.*

Then $\quad N(T) = \{x : x \in U, T(x) = \mathbf{0}\}$

is a sub-space of U (F).

Proof. Here $\mathrm{N}(\mathrm{T}) = \{x : x \in \mathrm{U}, \mathrm{T}(x) = \mathbf{0}\}$.

Clearly $\quad \mathrm{N}(\mathrm{T}) \subset \mathrm{U}$.

Now for $x, y \in \mathrm{N}(\mathrm{T}) \;\Rightarrow\; \mathrm{T}(x) = \mathbf{0}, \mathrm{T}(y) = \mathbf{0}$ ...(1)

Also $x, y \in \mathrm{N}(\mathrm{T}) \;\Rightarrow\; x, y \in \mathrm{U}$

$\Rightarrow\; \alpha x + \beta y \in \mathrm{U}$ for $\alpha, \beta \in \mathrm{F}$

$\therefore\quad \mathrm{T}(\alpha x + \beta y) = \alpha\,\mathrm{T}(x) + \beta\,\mathrm{T}(y)$ [*By Linearity of T*]

$= \alpha \,.\, \mathbf{0} + \beta \,.\, \mathbf{0}$ [$\because$ *of (1)*]

$= \mathbf{0}$.

$\Rightarrow\quad \alpha x + \beta y \in \mathrm{N}(\mathrm{T})$.

Hence N (T) is a sub-sapce of U.

6. Rank and Nullity

(*i*) **Rank. Def.** *Let U (F) and V (F) be two vector spaces and T : U → V be a linear transformation, then the rank of T is a dimension of the range space of T.*

This is usually denoted by ρ (T).

Symbolically. ρ (T) = dimension [Range (T)].

(*ii*) **Nullity. Def.** *Let U (F) and V (F) be two vector spaces and T : U → V be a linear transformation, then the nullity of T is the dimension of null space of T.*

This is usually denoted by υ (T).

Symbolically. υ (T) = dim. [Null space of (T)].

Theorem. *If T be a linear transformation on an n-dimensional vector space V (F), then*

$$\rho\,(T) + \upsilon\,(T) = n$$

i.e., Rank (T) + Nullity (T) = dim. (V). **(G.N.D.U. 1997 ; Pbi. U. 1997, 96, 85)**

Proof. Since N (T) ⊂ V (F) and V (F) is finite dimensional, so let

$$B_1 = \{x_1, x_2, \ldots, x_k\} \subset V, \text{ where } k \le n$$

is the basis set of N (T). Also we can extend this basis set to the basis set of V (F) and let after extending

$$B_2 = \{x_1, x_2, \ldots\ldots, x_k, k_{k+1}, \ldots\ldots, x_n\}$$

be the basis set of V (F).

Consider the set

$$B_3 = \{T(x_{k+1}), T(x_{k+2}), \ldots\ldots, T(x_n)\}$$

and show it to be the basis set of R (T).

(*i*) **To prove** : B_3 *is L.I.*

Let $\sum_{i=k+1}^{n} \alpha_i T(x_i) = 0 \in W$ for each $\alpha_i \in F$ and $x_i \in V$

$\Rightarrow$ $\Sigma\, T(\alpha_i x_i) = 0$ [*By linearity of T*]

$\Rightarrow$ $T(\Sigma\, \alpha_i x_i) = 0$ [*By linearity of T*]

$\Rightarrow$ $\Sigma\, \alpha_i x_i \in N(T)$. [$\because$ $\Sigma\, \alpha_i x_i \in V$]

Now $\sum_{i=k+1}^{n} \alpha_i x_i = \sum_{j=1}^{k} \beta_j x_j \;\forall \beta_j \in F$.

[$\because$ *each element of N (T) is linear combination of elements of* B_1]

$\Rightarrow$ $\sum_i (-\alpha_i) x_i + \sum_j \beta_j x_j = 0$

$\Rightarrow$ (*i.e.*, of elements of B_2) = 0

$\Rightarrow$ each $\alpha_i = 0$ and each $\beta_j = 0$.

Thus set B_3 is L.I.

(*ii*) **To prove** : $R(T) = L(B_3)$.

For any vector $y \in R(T)$ $\exists$ $x \in V$ such that $T(x) = y$ ...(1)

Since B_3 is basis of V and $x \in V$,

$\therefore$ $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k + \ldots + \alpha_n x_n$ for α_i's $\in F$.

So by (1), we have

$$y = T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k + \ldots + \alpha_n x_n)$$
$$= \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_k T(x_k) + \alpha_{k+1} T(x_{k+1}) + \ldots + \alpha_n T(x_n) \quad \text{[By linearity of } T]$$
$$= \alpha_1 0 + \alpha_2 0 + \ldots + \alpha_k 0 + \alpha_{k+1} T(x_{k+1}) + \ldots + \alpha_n T(x_n)$$

[Since $x_1, x_2, \ldots, x_k \in N(T)$, so $T(x_1) = 0$, $T(x_2) = 0, \ldots, T(x_k) = 0$]

$$= \alpha_{k+1} T(x_{k+1}) + \alpha_{k+2} T(x_{k+2}) + \ldots + \alpha_n T(x_n).$$

Thus B_3 linearly spans R (T), *i.e.*, R (T) = L (B_3).

Hence by (*i*) and (*ii*), B_3 is basis of R (T) and as number of elements in B_2 is $n - k$ so dimension of R (T) is $n - k$, *i.e.*, $\rho(T) = n - k$.

Also since B_1 is basis of N (T) so $\upsilon(T) = k$.

Thus we have $(n - k) + k = n$.

Hence $\rho(T) + \upsilon(T) = n$.

SOLVED EXAMPLES

Example 1. *For each of the following linear mappings T, find a basis and the dimension of (a) its range, (b) its null space.*

Also verify Rank (T) + Nullity (T) = dimension V.

(*i*) $T : R^4 \to R^3$ *defined by*

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4)$$

(*ii*) $T : R^3 \to R^2$ *defined by* $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$

(*iii*) $T : R^2 \to R^3$ *defined by* $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$

(*iv*) $T : R^2 \to R^3$ *defined by* $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$

(*v*) $T : R^3 \to R^3$ *defined by* $T(x_1, x_2, x_3) = (x_1 + 2x_2, x_2 - x_3, x_1 + 2x_3)$ (Pbi. U. 1986)

(*vi*) $T : R^3 \to R^2$ *defined by* $T(x, y, z) = (y + z, x + y - 2z, x + 2y - 2z)$ (Pbi. U. 1990)

(*vii*) $T : R^3 \to R^3$ *defined by* $T(x, y, z) = (3x, x - y, 2x + y + z)$. (P. U. 1996)

Sol. (*i*) We know that the set A = $\{e_1, e_2, e_3, e_4\}$ is a basis set for R^4,

where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$.

By def., $T(e_1) = T(1, 0, 0, 0) = (1 - 0 + 0 + 0, 1 + 2(0) - 0, 1 + 0 + 3(0) - 3(0))$
$= (1, 1, 1)$.

Similarly $T(e_2) = (-1, 0, 1)$, $T(e_3) = (1, 2, 3)$

and $T(e_4) = (1, -1, -3)$ [*Verify !*]

Now for any $x \in R^4 = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$ as A is basis.

$\therefore$ $y \in R(T) = T(x) = T(a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4)$

$= a_1 T(e_1) + a_2 T(e_2) + a_3 T(e_3) + a_4 T(e_4)$ [*By Linearity of T*]

$= a_1 (1, 1, 1) + a_2 (-1, 0, 1) + a_3 (1, 2, 3) + a_4 (1, -1, -3)$.

To verify whether $y \in R(T)$ expressed as linear combination of four vectors $\in R^3$ can be expressed as a linear combination of fewer number of vectors or not.

For this, we compute a matrix whose rows are these four vectors

$$B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \qquad [\textit{Operating } R_1 \to R_2 + R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_3 \to R_3 - R_2, R_4 \to R_4 + 2R_3]$$

which is echelon form of matrix.

Thus the set of non-zero vectors {(1, 1, 1), (0, 1, 2)} is the basis for R (T).

Hence dimension R (T) *i.e.,* rank (T) = 2.

To find the basis and dimension for N (T).

$x \in$ N (T) if T $(x) = 0$.

Now T $(x_1, x_2, x_3, x_4) = 0$

$\Rightarrow$ $(x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4) = (0, 0, 0)$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_3 - x_4 &= 0 \\ x_1 + x_2 + 3x_3 - 3x_4 &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

$$\text{Co-efficient matrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \qquad [\textit{Operating } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_3 \to R_3 - 2R_2]$$

which is echelon form of matrix.

Thus the system (1) is equivalent to

$x_1 - x_2 + x_3 + x_4 = 0$ ...(2)

$x_2 + x_3 - 2x_4 = 0$ ...(3)

From (3), $x_2 = -x_3 + 2x_4$

Putting in (2), $x_1 + x_3 - 2x_4 + x_3 + x_4 = 0$

$\Rightarrow$ $x_1 = -2x_3 + x_4$

Thus $x_1 = -2x_3 + x_4$

$x_2 = -x_3 + 2x_4.$

Here x_3 and x_4 are free variables.

Hence nullity T = dimension N (T) = No. of free variables = 2.

Choosing $x_3 = 1, x_4 = 0, x_1 = -2, x_2 = -1$.

Choosing $x_3 = 0, x_4 = 1, x_1 = 1, x_2 = 2$.

$\{(-2, -1, 1, 0), (1, 2, 0, 1)\}$ constitutes a basis for N (T) because the above system is L.I. also.

Now dimension $R^4 = 4$, dimension R (T) = 2, dimension N (T) = 2

$\therefore$ Rank (T) + Nullity (T) = dimension R^4.

Other forms

(I) Let $T : R^4 \to R^3$ be a linear mapping defined by

$$T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t).$$

Find the dimensions of

(*a*) Range of T (*b*) Null space of T, and verify that

Rank (T) + Nullity (T) = dimension (R^4). *(Pbi. U. 1990 ; P.U. 1987, 85 ; G.N.D.U. 1986)*

(II) Let $T : R^4 \to R^3$ be a linear mapping defined by

$$T(x, y, z, u) = (x - y + z + u, x + 2z - u, x + y + 3z - 3u).$$

Find the dimensions of

(*a*) Range of T (*b*) Null space of T, and verify that

Rank (T) + Nullity (T) = dimension (R^4). *(P.U. 1985)*

(*ii*) We know that the set A = $\{e_1, e_2, e_3\}$ is a basis set for R^3, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

By def., $T(e_1) = T(1, 0, 0) = (1 + 0, 0 + 0) = (1, 0)$

$T(e_2) = T(0, 1, 0) = (0 + 1, 1 + 0) = (1, 1)$

and $T(e_3) = T(0, 0, 1) = (0 + 0, 0 + 1) = (0, 1)$.

Now for any $x \in R^3 = a_1 e_1 + a_2 e_2 + a_3 e_3$ as A is basis.

$$\therefore \quad y \in R(T) = T(x) = T(a_1 e_1 + a_2 e_2 + a_3 e_3)$$

$$= a_1 T(e_1) + a_2 T(e_2) + a_3 T(e_3) \qquad \textit{[By Linearity of T]}$$

$$= a_1 (1, 0, 0) + a_2 (0, 1, 0) + a_3 (0, 0, 1).$$

To verify, whether $y \in R(T)$ expressed as linear combination of three vectors $\in R^2$ can be expressed as a linear combination of fewer number of vectors or not.

For this, we compute a matrix whose rows are these three vectors.

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad [\textit{Operating } R_2 - R_1]$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_3 - R_2]$$

which is echelon form of matrix.

Thus the set of non-zero vectors $\{(1, 0), (0, 1)\}$ is the basis for R (T).

Hence dimension R (T) *i.e.*, rank (T) = 2.

Now it is same as part (*i*).

(*iii*) We know that the set A = $\{e_1, e_2\}$ is a basis set for R^2, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

By def., $\quad T(e_1) = T(1, 0) = (1 + 0, 1 - 0, 0) = (1, 1, 0)$

and $\quad T(e_2) = T(0, 1) = (0 + 1, 0 - 1, 1) = (1, -1, 1)$.

Now for any $x \in R^2 = a_1 e_1 + a_2 e_2$ as A is basis

$\therefore \quad y \in R(T) = T(x) = T(a_1 e_1 + a_2 e_2)$

$= a_1 T(e_1) + a_2 T(e_2)$ [*By linearity of T*]

$= a_1 (1, 1, 0) + a_2 (1, -1, 1)$.

To verify whether $y \in R(T)$ expressed as linear combination of two vectors $\in R^3$ can be expressed as a linear combination of fewer number of vectors or not.

For this, we compute a matrix whose rows are these two vectors.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \text{ which is not echelon form of matrix.}$$

Thus $\{(1, 1, 0), (1, -1, 1)\}$ is the basis for R (T).

Hence dimension R (T), *i.e.*, rank (T) = 2.

To find the basis and dimension for N (T).

$x \in N(T)$ if $T(x) = 0$.

Now $\quad T(x_1, x_2) = 0$

$\Rightarrow \quad (x_1 + x_2, x_1 - x_2, x_2) = (0, 0, 0)$

$\Rightarrow \quad x_1 + x_2 = 0, x_1 - x_2 = 0, x_2 = 0$

$\Rightarrow \quad x_1 = 0, x_2 = 0$

$\therefore \quad (x_1, x_2) = (0, 0) = \mathbf{0} \in R^2$.

Hence the null space consists of only zero vector.

$\therefore \quad N(T) = 0$.

Now dimension $R^2 = 2$, dimension R (T) = 2, dimension N (T) = 2

$\therefore$ Rank (T) + Nullity (T) = dimension R^2.

(*iv*) We know that the set A = $\{e_1, e_3\}$ is a basis set for R^2, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

By def., $T(e_1) = T(1, 0) = (1 - 0, 0 - 1, -1) = (1, -1, -1)$

and $\quad T(e_2) = T(0, 1) = (0 - 1, 1 - 0, 0) = (-1, 1, 0)$.

Now for $x \in R^2 = a_1 e_1 + a_2 e_2$ as A is basis

$\therefore \quad x \in R(T) = T(x) = T(a_1 e_1 + a_2 e_2)$

$= a_1 T(e_1) + a_2 T(e_2)$ [*By Linearity of T*]

$= a_1 (1, -1, -1) + a_2 (-1, 1, 0)$.

To verify whether $y \in R(T)$ expressed as a linear combination of two vectors $\in R^3$ can be expressed as combination of fewer number of vectors or not.

For this, we compute a matrix whose rows are these two vectors

$$B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \text{ which is not echelon form of matrix.}$$

Thus $\{(1, -1, -1), (-1, 1, 0)\}$ is the basis for R (T).

Hence dimension R (T), *i.e.*, rank (T) = 2.

To find the basis and dimension for N (T).

$x \in N(T)$ if $\quad T(x) = \mathbf{0}$.

Now $\quad T(x_1, x_2) = 0$

$(x_1 + x_2, x_2 - x_1, -x_1) = (0, 0, 0)$

$\Rightarrow \quad x_1 + x_2 = 0, x_2 - x_1 = 0, -x_1 = 0$

$\Rightarrow \quad x_1 = 0, x_2 = 0$

$\Rightarrow \quad (x_1, x_2) = (0, 0) = 0 \in R^2.$

Hence the null space consists of only zero vector.

$\therefore \quad N(T) = 0.$

Now dimension $R^2 = 2$, dimension R (T) = 2, dimension N (T) = 2

$\therefore \quad$ Rank (T) + Nullity (T) = dimension R^3.

(*v*) We know that the set $A = \{e_1, e_2, e_3\}$ is a basis set for R^3, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

By def., $T(e_1) = T(1, 0, 0)$

$= (1 + 2 . 0, 0 - 0, 1 + 2 . 0) = (1, 0, 1),$

$T(e_2) = T(0, 1, 0)$

$= (0 + 2 . 1, 1 - 0, 0 + 2 . 0) = (2, 1, 0)$

and $\quad T(e_3) = T(0, 0, 1)$

$= (0 + 2 . 0, 0 - 1, 0 + 2 . 1) = (0, -1, 2).$

Now for any $x \in R^3 = a_1 e_1 + a_2 e_2 + a_3 e_3$ as A is basis.

$\therefore \quad y \in R(T) = T(x) = T(a_1 e_1 + a_2 e_2 + a_3 e_3)$

$= a_1 T(e_1) + a_2 T(e_2) + a_3 T(e_3)$ *[By Linearity of T]*

$= a_1 (1, 0, 1) + a_2 (2, 1, 0) + a_3 (0, -1, 2).$

To verify whether $y \in R(T)$ expressed as a linear combination of three vectors $\in R^3$ can be expressed as a linear combination of fewer number of vectors or not.

For this, we compute a matrix where rows are these three vectors

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \qquad [\textit{Operating } R_2 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_3 + R_2]$$

which is echelon form of matrix.

Thus the set of non-zero vectors $\{(1, 0, 1), (0, 1, -2)\}$ is the basis for R (T).

Hence dimension R (T) *i.e.*, rank (T) = 2.

To find the basis and dimension for N (T).

$x \in N(T)$ if $T(x) = 0$

Now $T(x_1, x_2, x_3) = 0$

$\Rightarrow \quad (x_1 + 2x_2, x_2 - x_3, x_1 + 2x_3) = (0, 0, 0)$

$\Rightarrow \quad a = x_1, 2a + b = x_2$

$\Rightarrow \quad a = x_1, b = x_2 - 2x_1$

$\Rightarrow \quad (x_1, x_2) = x_1 (1, 2) + (x_2 - 2x_1)(0, 1)$

Thus (1, 2) and (0, 1) span R^2.

Hence $T(x_1, x_2) = T[x_1 (1, 2) + (x_2 - 2x_1)(0, 1)] = x_1 T(1, 2) + (x_2 - 2x_1) T(0, 1)$

$= x_1 (3, -1, 5) + (x_2 - 2x_1)(2, 1, -1)$

$= (2x_2 - x_1, x_2 - 3x_1, 6x_1 - x_2),$

which is the required L.T.

(*b*) **Null space of T**

Let $x = (x_1, x_2)$, then $N(T) = \{x \in R^3 : T(x) = 0' \in R^3\}$

But $T(x) = (2x_2 - x_1, x_2 - 3x_1, 6x_1 - x_2) = 0' = (0, 0, 0)$

$\Rightarrow \quad 2x_2 - x_1 = 0, x_2 - 3x_1 = 0, 6x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2 = 0$

$\therefore \quad x = (x_1, x_2) = (0, 0) = 0 \in R^2$

Thus null space consists of only zero vector of the domain.

Range space of T

Range space of $T = \{y \in R^3 : y = T(x)$ for some $x \in R^2\}$

Thus range space of T consists of all triplets of real, x_1, x_2 such that $(x_1, x_2) \in R^2$.

Nullity T

Nullity T = dimension of null space = 0.

Rank of T : In order to find **Rank T**, we should find basis for R (T) and the number of elements in the basis set = Rank T.

$$y \in R(T) = T(x) = T(x_1, x_2) = (2x_2 - x_1, x_2 - 3x_1, 6x_1 - x_2) \quad \ldots(1)$$

Now $(x_1, x_2) \in V_2(R) = x_1 (1, 0) + x_2 (0, 1)$

$= x_1 e_1 + x_2 e_2$, where $e_1, e_2 \in V_2(R)$

$\therefore \quad y = T(x_1, x_2) = T(x_1 e_1 + x_2 e_2) = x_1 T(1, 0) + x_2 T(0, 1)$

$= x_1 (-1, -3, 6) + x_2 (2, 1, -1)$

Since $y \in R(T) = x_1 (-1, -3, 6) + x_2 (2, 1, -1)$,

$\therefore$ the vectors $(-1, -3, 6)$ and $(2, 1, -1)$ span R (T).

To check whether these are L.I.

Consider $a(-1, -3, 6) + b(2, 1, -1) = 0' = (0, 0, 0)$

$\Rightarrow \quad (-a + 2b, -3a + b, 6a - b) = (0, 0, 0)$

$\Rightarrow \quad a = b = 0.$

Thus L.I.

Hence $(-1, -3, 6), (2, 1, -1)$ is the basis set of two vectors and dimension (T) = rank (T) = 2.

Example 4. (*i*) *Find a linear transformation $T : R^3 \to R^3$ whose range is generated by $(1, 0, -1)$ and $(1, 2, 2)$.* (G.N.D.U. 1995 S)

(*ii*) *Find a linear transformation $T : R^3 \to R^4$ whose range space is generated by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$.* (G.N.D.U. 1996)

Sol. (*i*) The basis of R^3 is $A = \{e_1, e_2, e_3\}$,

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Let $T(1, 0, 1) = (1, 0, -1), T(0, 1, 0) = (1, 2, 2)$ and $T(0, 0, 1) = (0, 0, 0)$

R (T) is generated by $T(e_i), i = 1, 2, 3$.

For each $(x_1, x_2, x_3) \in R^3$, we have

$$(x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3 (0, 0, 1)$$

$$\Rightarrow \quad T(x_1, x_2, x_3) = x_1 T(1, 0, 0) + x_2 T(0, 1, 0) + x_3 (0, 0, 1)$$

$$= x_1 (1, 0, -1) + x_2 (1, 2, 2) + x_3 (0, 0, 0)$$

Hence $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_2, -x_1 + 2x_2)$, which is the required L.T.

(*ii*) The basis of R^3 is $A = \{e_1, e_2, e_3\}$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Let $T(1, 0, 0) = (1, 2, 0, -4), T(0, 1, 0) = (2, 0, -1, -3)$ and $T(0, 0, 1) = (0, 0, 0, 0)$

R (T) is generated by $T(e_i)$, $i = 1, 2, 3$.

For each $(x_1, x_2, x_3) \in R^3$, we have

$$(x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3 (0, 0, 1)$$

$$\Rightarrow \quad T(x_1, x_2, x_3) = x_1 T(1, 0, 0) + x_2 T(0, 1, 0) + x_3 (0, 0, 1)$$

$$= x_1 (1, 2, 0, -4) + x_2 (2, 0, -1, -3) + x_3 (0, 0, 0, 0)$$

Hence $T(x_1, x_2, x_3) = (x_1 + 2x_2, 2x_1, -x_2, -4x_1 - 3x_2)$, which is the required L.T.

Example 5. (*i*) *Find a linear transformation $T : R^3 \to R^3$ whose image is generated by (1, 2, 3) and (4, 5, 6).*

(*ii*) *Find a linear transformation $T : R^4 \to R^3$ whose null space is generated by (1, 2, 3, 4) and (0, 1, 1, 1).* (Pbi. U. 1987 ; G.N.D.U. 1985)

(*iii*) *Find a linear transformation $T : R^4 \to R^3$ whose null space is generated by $S = \{(2, 3, 4, 1), (1, 0, 1, 1)\}$.* (G.N.D.U 1989 ; P.U. 1987)

Sol. (*i*) The basis of R^3 is $A = \{e_1, e_2, e_3\}$,

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Let $T(1, 0, 0) = (1, 2, 3), T(0, 1, 0) = (4, 5, 6)$ and $T(0, 0, 1) = (0, 0, 0)$

R (T) is generated by $T(e_i)$; $i = 1, 2, 3$.

For each $(x_1, x_2, x_3) \in R^3$, we have

$$(x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3 (0, 0, 1)$$

$$\Rightarrow \quad T(x_1, x_2, x_3) = x_1 T(1, 0, 0) + x_2 T(0, 1, 0) + x_3 T(0, 0, 1)$$

$$= x_1 (1, 2, 3) + x_2 (4, 5, 6) + x_3 (0, 0, 0)$$

$$= (x_1 + 4x_2, 2x_1 + 5x_2, 3x_1 + 6x_2),$$

which is the required L.T.

(*ii*) Let K be the null space of T.

Since K is generated by $v_1 = (1, 2, 3, 4)$ and $v_2 = (0, 1, 1, 1)$ and v_1 is not a scalar multiple of v_2,

$\therefore$ these are L.I. over R, and dim. K = 2.

$$\begin{aligned}\therefore \quad T(x, y, z, t) &= aT((2, 3, 4, 1)) + bT((1, 0, 1, 1)) + cT((0, 0, 1, 0)) + dT(0, 0, 0, 1)\\ &= a\,(0, 0, 0) + b\,(0, 0, 0) + c\,(1, 0, 0) + d\,(0, 1, 0)\\ &= (c, d, 0)\\ &= \left(z - x - \frac{2y}{3},\, t - x + \frac{y}{3}, 0\right).\end{aligned}$$

Example 6. *Describe explicitly a linear transformation from R^3 to R^3 which has as its range the sub-space spanned by $(1, 0, -1)$ and $(1, 2, 2)$.*

Sol. The basis of R^3 is $A = \{e_1, e_2, e_3\}$,
where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Let $T(1, 0, 0) = (1, 0, -1)$, $T(0, 1, 0) = (1, 2, 2)$ and $T(0, 0, 1) = (0, 0, 0)$.

$R(T)$ is generated by $T(e_i)$, $i = 1, 2, 3$.

For each $(x_1, x_2, x_3) \in R^3$, we have

$$(x_1, x_2, x_3) = x_1\,(1, 0, 0) + x_2\,(0, 1, 0) + x_3\,(0, 0, 1)$$

$$\begin{aligned}\Rightarrow \quad T(x_1, x_2, x_3) &= x_1\,T(1, 0, 0) + x_2\,T(0, 1, 0) + x_3\,T(0, 0, 1)\\ &= x_1\,(1, 0, -1) + x_2\,(1, 2, 2) + x_3\,(0, 0, 0)\\ &= (x_1 + x_2, 2x_2, -x_1 + 2x_3),\end{aligned}$$

which is the required L.T.

Example 7. (*a*) *Let V be a vector space of 2×2 matrix over R and let*

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

Let $T : V \to V$ be the linear map defined by $T(A) = AM - MA$. Find the basis and dimension of the null space of T.

(*b*) *Let V be the vector space of 2×2 matrices over R and let*

$$M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

Let $T : V \to V$ be the linear map defined by $T(A) = MA \;\; \forall A \in V$. Find the basis and dimension of:
(*i*) *Null space of T,* (*ii*) *Range of T.*

Sol. (*a*) Here we are to find $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that

$$T\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \ldots(1)$$

By def.,

$$\begin{aligned}T\begin{bmatrix} x & y \\ z & t \end{bmatrix} &= \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix}\\ &= \begin{bmatrix} x & 2x + 3y \\ z & 2z + 3t \end{bmatrix} - \begin{bmatrix} x + 2z & y + 2t \\ 3z & 3t \end{bmatrix}\\ &= \begin{bmatrix} x - x - 2z & 2x + 3y - y - 2t \\ z - 3z & 2z + 3t - 3t \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} -2z & 2x+2y-2t \\ -2z & 2z \end{bmatrix}$$

$\therefore$ From (1), $\begin{bmatrix} -2z & 2x+2y-2t \\ -2z & 2z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad -2z = 0, \quad 2x + 2y - 2t = 0$

$\Rightarrow \quad z = 0, \quad x + y - t = 0$

$\Rightarrow \quad x = -y + t, \quad z = 0.$

Here y and t are free variables.

Hence dim. (N (T)) = 2.

For basis of Null space of T :

(I) For $y = -1,\ t = 0$, we get

$x = 1,\ y = -1,\ z = 0,\ t = 0.$

(II) For $y = 0,\ t = -1$, we get

$x = 1,\ y = 0,\ z = 0,\ t = 1.$

Hence $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is the basis of null space of T.

(*b*) (*i*) As in part (*a*), the basis set will be

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Under T, the image of element of B generates the range of R (T) of T.

By def. the generators of R (T) are

$$\left.\begin{aligned}
T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \\
T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \\
T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \\
T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}
\end{aligned}\right\} \quad \ldots(1)$$

Now we form the matrix whose rows are elements of generators (1) of R (T)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\textit{Operating } R_3 + R_1 \textit{ and } R_4 + R_2]$$

which is of echelon form.

Thus $\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$ is the basis set of R (T).

Hence R (T) = 2.

(*ii*) The null space of T is the set of all 2 × 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that T $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad [\textit{By def.}]$$

$$\Rightarrow \begin{bmatrix} a - c & b - d \\ 2a + 2c & -2b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus we get

$$a - c = 0, \quad b - d = 0, \quad -2a + 2c = 0 \text{ and } -2b + 2d = 0$$

$$\Rightarrow \quad a - c = 0, \quad b - d = 0$$

$\therefore$ c, d are free variables.

Hence dim. $(\text{N (T)}) = 2$.

When $c = 1, \ d = 0,$

we get the solution $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

When $c = 0, \ d = 1,$

we get the solution $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Hence the basis set N (T) of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Example 8. *Let V be a vector space and T a linear transformation from V. Prove that the following two statements are equivalent :*

(*i*) *The intersection of range of T and null space of T is the zero sub-space of T.*

(*ii*) *If* $T(T\alpha) = \mathbf{0}$, *then* $T(\alpha) = \mathbf{0}$.

Sol. Given : R (T) ∩ N (T) = {**0**}.

To prove : $T(T\alpha) = \mathbf{0} \Rightarrow T(\alpha) = \mathbf{0}$.

Put $T\alpha = \beta$, by def. of R (T), $\beta \in$ R (T) ...(1)

Also $T(T\alpha) = 0$ [*Given*]

$\Rightarrow$ $T(\beta) = 0$ $\Rightarrow \beta \in N(T)$ ...(2)

(1) and (2) $\Rightarrow \beta \in R(T) \cap N(T)$

$\Rightarrow \beta \in \{0\}$ $\Rightarrow \beta = 0$

$\Rightarrow T(\alpha) = 0$

Hence (*i*) $\Rightarrow$ (*ii*).

Given : $T(T\alpha) = 0 \Rightarrow T(\alpha) = 0$.

To prove : $R(T) \cap N(T) = \{0\}$.

Let $\beta \in R(T) \cap N(T)$

$\therefore$ $\beta \in R(T)$ and $\beta \in N(T)$

Now $\beta \in R(T) \Rightarrow \beta = T(\alpha)$ for some $\alpha \in V$

and $\beta \in N(T) \Rightarrow T(\beta) = 0 \Rightarrow T(T\alpha) = 0 \Rightarrow T(\alpha) = 0$ [*Given*]

$\Rightarrow \beta = 0$ as $\beta = T(\alpha)$.

Thus $R(T) \cap N(T) = \{0\}$

Hence (*ii*) $\Rightarrow$ (*i*).

Example 9. *Let V (F) be an n-dimensional vector space and let T be a linear transformation from V into V such that range and null space of T are identical. Prove that n is even.*

Sol. We know that

Rank (T) + Nullity (T) = dim. V = n ...(1)

Since R (T) and N (T) are identical, [*Given*]

$\therefore$ dim. R (T) = dim. N (T)

$\Rightarrow$ Rank (T) = Nullity (T)

From (1), Rank T + Rank (T) = n

$\Rightarrow$ 2 Rank (T) = n $\Rightarrow$ n is even.

Example 10. *Let V be a finite dimensional vector space and T a linear operator on V such that rank T^2 = rank T. Prove that the range and null space of T are disjoint.* (G.N.D.U. 1987)

Sol. $T : V \to V \Rightarrow T^2 : V \to V$,

and Rank T + Nullity T = dim. V ...(1)

and Rank T^2 + Nullity T^2 = dim. V ...(2)

(1) and (2) $\Rightarrow$ Rank T + Nullity T = Rank T^2 + Nullity T^2

$\Rightarrow$ Nullity T = Nullity T^2 [$\because$ *Rank T^2 = Rank T (given)*]

$\Rightarrow$ dim. (null space of T) = dim. (null space of T^2) ...(3)

Now $\alpha \in$ null space of T $\Rightarrow T(\alpha) = 0$

$\Rightarrow$ $T[T(\alpha)] = T(0) \Rightarrow T^2(\alpha) = 0$

$\Rightarrow \alpha \in$ null space of T^2

$\Rightarrow$ null space of T $\subset$ null space of T^2 ...(4)

From (1), dimensions are equal

$\therefore$ null space of T = null space of T^2

$\therefore\ \alpha \in$ null space of $T^2 \Rightarrow \alpha \in$ null space of T

$\therefore\ T^2(\alpha) = 0 \Rightarrow T(\alpha) = 0$

$\therefore\ R(T) \cap N(T) = \{0\}$.

Hence Range of T and null space of T are disjoint.

Example 11. *What do you understand by invariant sub-space ? Hence prove that range and space of a L.T.T on V are invariant under T.*

Sol. [**Def.** Let $T : V \to V$. Let W be the sub-space of V, then W is invariant under T if $\forall\ \alpha \in W \Rightarrow T(\alpha) \in W$]

$R(T) = \{\beta : \beta = T(\alpha)$ for some $\alpha \in V\}$

We know that R (T) is a sub-space of T because $T : V \to V$

$\therefore\quad \beta \in R(T) \Rightarrow \beta \in V$

and $\quad \beta \in R(T) \Rightarrow T(\beta) \in R(T)$

Hence R (T) is invariant under T.

Now $N(T) = \{\alpha : \alpha \in V, T(\alpha) = 0\}$

We know that N (T) is a sub-space of V.

Since $T : V \to V$ and consequently it contains zero vector

$\therefore\quad \alpha \in N(T) \Rightarrow T(\alpha) = 0$ and $0 \in N(T)$

Since N (T) is a sub-space,

$\therefore\quad \alpha \in N(T) \Rightarrow T(\alpha) \in N(T)$.

Hence N (T) is invariant under T.

7. Singular and Non-Singular Transformations

(*i*) **Singular Transformation. Def.** *A linear transformation $T : U \to V$ is called singular if the null space of T consists of at least one non- zero vector.*

Remember : *T is singular if $\alpha \neq 0 \Rightarrow T(\alpha) = 0'$ for some $\alpha \in U$.*

(*ii*) **Non-Singular Transformation. Def.** *A linear transformation $T : U \to V$ is called non-singular if the null space of T is the zero space* $\{0\}$, *i.e., it consists of only the zero element.*

Remember : *If $T(\alpha) = 0' \Rightarrow \alpha = 0$, then T is non-singular.*

If $\alpha \neq 0 \Rightarrow T(\alpha) \neq 0'$, then T is non-singular.

8. Injective, Surjective and Bijective Transformations

(*i*) **Injective Transformation. Def.** *The transformation $T : U \to V$ is called injective if distinct elements of the domain have distinct images.*

This is also called **one-one transformation.**

Symbolically : $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$

$T(x_1) \neq T(x_2) \Rightarrow x_1 \neq x_2$.

(*ii*) **Surjective Transformation. Def.** *The transformation $T : U \to V$ is called surjective if each element $y \in V$ is the T image of some element $x \in U$.*

This is also called **onto-transformation.**

Symbolically : $y \in V \Rightarrow \exists\, x \in U$ s.t. $T(x) = y$.

Thus $R(T) = V$.

(*iii*) **Bijective Transformation. Def.** *The transformation $T : U \to V$ is called bijective if it is injective as well as surjective.*

This is also called **one-one and onto-transformation.**

(*iv*) **Isomorphism of vector space.** *Let $U(F)$ and $V(F)$ be two vector spaces over a field F and $T : U \to V$ be a linear transformation, then it is said to be an isomorphism of U onto V if T is one-one and onto.*

Then the two vector spaces U and V are said to be isomorphic and we write it as $U \simeq V$.

THEOREMS

Theorem I. *Two finite dimensional vector spaces U(F) and V(F) over the same field F are isomorphic iff they have the same dimensions, i.e.,*

$U \simeq V \Leftrightarrow \dim. U = \dim. V.$ (G.N.D.U. 1989)

Proof. Given. U(F) and V(F) be isomorphic.

(*i*) **To prove.** dim. U = dim. V.

Let T be an isomorphism from U to V, so that T is one-one and onto L.T. from U to V.

Since U is finite dimensional,

$\therefore$ it has a finite basis set.

Suppose $B_1 = \{u_1, u_2, \ldots\ldots, u_n\}$ is a basis of U, where dim. U = n.

Let the set of image vectors by T be

$$B_2 = \{T(u_1), T(u_2), \ldots\ldots, T(u_k)\},$$

which is a subset of V consisting of n elements.

If we prove that B_2 is a basis set of V, then dim. V = n.

For this, we shall prove tnat B_2 is L.I. and linear space of B_2 is V.

(*ii*) **To prove.** *B_2 is L.I.*

$$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots\ldots + \alpha_n T(u_n) = \mathbf{0}$$

$$\Rightarrow \quad T(\alpha_1 u_1) + T(\alpha_2 u_2) + \ldots\ldots + T(\alpha_n u_n) = T(\mathbf{0}) \qquad [\because T(\mathbf{0}) = \mathbf{0}]$$

$$\Rightarrow \quad T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n) = T(\mathbf{0}) \qquad [\because T \text{ is linear}]$$

$$\Rightarrow \quad \alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n = \mathbf{0}$$

$$\Rightarrow \quad \alpha_1 = 0,\ \alpha_2 = 0,\ \ldots\ldots, \alpha_n = 0 \qquad [\because B_1 \text{ is a basis, } \therefore B_1 \text{ is L.I.}]$$

Thus B_2 is L.I.

(*iii*) **To prove.** *B_2 spans V.*

Let $v \in V$.

Since T is onto, $\therefore$ there exists $u \in U$ s.t. $T(u) = v$.

But $u = \beta_1 u_1 + \beta_2 u_2 + \ldots\ldots + \beta_n u_n$ $\qquad [\because B_1 \text{ is a basis of } U]$

$$\therefore \quad v = T(u)$$

$$= T(\beta_1 u_1 + \beta_2 u_2 + \ldots\ldots + \beta_n u_n)$$

$$= \beta_1 T(u_1) + \beta_2 T(u_2) + \ldots\ldots + \beta_n T(u_n) \qquad [\because T \text{ is linear}]$$

$\therefore$ v is a linear combination of the elements of B_2.

Thus B_2 spans V.

Hence dim. $V = n \Rightarrow$ dim. U = dim. V.

Conversely : Given dim. U = dim. V = n.

To prove. $U \cong V$, *i.e.*, there exists an isomorphism between U and V.

Since dim. U = dim. V = n,

$\therefore$ there exist basis sets U and V, each having n elements.

Let $B_1 = \{u_1, u_2, \ldots\ldots, u_n\}$ and $B_2 = \{v_1, v_2, \ldots\ldots, v_n\}$ be the basis sets of U and V respectively.

$\therefore$ Each member of U and V can be expressed as a linear combination of the elements of B_1 and B_2 respectively.

$\therefore$ $u \in U \Rightarrow$ there exist scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

s.t. $$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n$$

Let us define $T : U \to V$ as

$$T(u) = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n .$$

We say that T is isomorphism,

$$T(u_i) = v_i \, ; \, i = 1, 2, \ldots\ldots, n.$$

(*i*) **To prove.** *T is linear.*

For $x, y \in U$ and $\alpha, \beta \in F$,

$$x = \sum_{i=1}^{n} \gamma_i u_i \text{ and } y = \sum_{i=1}^{n} \delta_i u_i$$

$$\therefore \qquad \alpha x + \beta y = \sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i)\, u_i$$

$$\therefore \qquad T(\alpha x + \beta y) = T\left(\sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i)\, u_i\right) = \sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i)\, v_i \qquad \textit{[By def.]}$$

$$= \sum_{i=1}^{n} \alpha\gamma_i v_i + \sum_{i=1}^{n} \beta\delta_i v_i = \alpha \sum_{i=1}^{n} \gamma_i v_i + \beta \sum_{i=1}^{n} \delta_i v_i$$

$$= \alpha T\left(\sum_{i=1}^{n} \gamma_i u_i\right) + \beta\left(\sum_{i=1}^{n} \delta_i u_i\right)$$

$$= \alpha T(x) + \beta\, T(y).$$

Thus T is linear.

(*ii*) **To prove.** *T is linear.*

Let $T(x) = T(y)$

$$\Rightarrow \quad \sum_{i=1}^{n} \gamma_i v_i = \sum_{i=1}^{n} \delta_i v_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} (\gamma_i - \delta_i)\, v_i = 0$$

$\Rightarrow \quad T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n) = 0'$ [$\because$ *T is linear*]

$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$ [$\because$ *of (1)*]

$\Rightarrow \quad \alpha_1 = 0,\ \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$ [$\because$ *S is L.I.*]

Hence S′ is L.I.

Conversely : Here T-image of any L.I. set is L.I.

To prove : T is non-singular.

Now $x \neq 0 \Rightarrow T(x) \neq 0'$

or $T(x) \neq 0 \Rightarrow x \neq 0$.

When $x \neq 0$ is a vector, then $\{x\}$ is a L.I. set and its T-image

i.e., $\{T(x)\}$ is also a L.I. and, hence, $T(x) \neq 0'$.

Hence T is non-singular.

Theorem V. *Let $T : U \to V$ be a linear transformation of U(F) into V(F). Suppose U(F) is finite dimensional. Prove that U and the range space of T have the same dimension iff T is non-singular.*

Proof. Let dim. U = dim. (Range T) = Rank (T) ...(1)

Since Rank (T) + Nullity (T) = dim. (U)

$\therefore$ Nullity (T) = 0 [$\because$ *of (1)*]

$\Rightarrow$ Null-space of T is zero-space {0}

Hence T is non-singular.

Conversely. Here T is non-singular.

Then the null-space consists of only zero element

$\Rightarrow$ Nullity (T) = 0 ...(2)

But Rank (T) + Nullity (T) = dim. (U)

$\therefore$ Rank (T) = dim. (U) [$\because$ *of (2)*]

Hence dim. (U) = dim. (Range T).

Theorem VI. *If U and V are finite-dimensional vector spaces of the same dimension, then a linear mapping $T : U \to V$ is one-one iff it is onto.*

Proof. T in one-one $\Leftrightarrow N(T) = \{0\}$

$\Leftrightarrow \upsilon(T) = 0$

$\Leftrightarrow \rho(T) + \upsilon(T) = \dim. U = \dim. V$

$\Leftrightarrow \rho(T) = \dim. U = \dim. V$

$\Leftrightarrow R(T) = V$

$\Leftrightarrow$ T is onto.

ALGEBRA OF LINEAR TRANSFORMATIONS

Theorem I. *The set L(U, V) of all linear transformations from U(F) into V(F) is a vector space over the field F with addition and scalar multiplication defined by*

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)\ \forall x \in U \text{ and } T_1, T_2 \in L(U, V)$$

$$(\alpha T_1)(x) = \alpha T_1(x)\ \forall x \in U, T_1 \in L(U, V) \text{ and } \alpha \in F.$$

Proof. In order to prove that L(U, V) is a vector space, we are to verify all the properties of a vector space.

I. Under Addition :

V_1. Closure Property. Let $T_1 : U \to V$ and $T_2 : U \to V$ be two linear transformations.

To prove : $T_1 + T_2$ defined by

$(T_1 + T_2)x = T_1(x) + T_2(x)\ \forall\, x \in U$ is also a L.T.

Since T_1 and T_2 are linear transformations $\in L(U, V)$,

$\therefore\quad T_1(x), T_2(x) \in V\ \forall\, x \in U$

$\Rightarrow\quad T_1(x) + T_2(x) \in V$ [$\because$ *V is a vector space*]

$\Rightarrow\quad T_1 + T_2 : U \to V$

Also for $x, y \in U$ and $\alpha, \beta \in F \quad \Rightarrow \quad \alpha x + \beta y \in U$

so $\quad (T_1 + T_2)(\alpha x + \beta y) = T_1(\alpha x + \beta y) + T_2(\alpha x + \beta y) = \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y)$

$$= \alpha\,(T_1(x) + T_2(x)) + \beta\,(T_1(y) + T_2(y)) = \alpha\,(T_1 + T_2)\,x + \beta\,(T_1 + T_2)\,y.$$

Thus $T_1 + T_2$ is a linear transformation from $U \to V$.

Hence the verification, *i.e.*,

$T_1, T_2 \in L(U, V) \quad \Rightarrow \quad T_1 + T_2 \in L(U, V).$

V_2. Associative Property. $\forall\, T_1, T_2, T_3 \in L(U, V)$, we have

$$((T_1 + T_2) + T_3)(x) = (T_1 + T_2)(x) + T_3(x) \qquad \forall\, x \in U$$
$$= (T_1(x) + T_2(x)) + T_3(x)$$
$$= T_1(x) + (T_2(x) + T_3(x)) \qquad \text{[By Associativity in } V\text{]}$$
$$= T_1(x) + (T_2 + T_3)(x) = (T_1 + (T_2 + T_3))(x).$$

Hence the verification.

V_3. Existence of Identity.

First of all, we define a zero mapping $O : U \to V$ as

$$O(x) = \mathbf{0} \quad \forall\ x \in U$$

For $x, y \in U$ and $\alpha, \beta \in F$,

$$O(\alpha x + \beta x) = \mathbf{0} \qquad \text{[By def.]}$$
$$= \alpha\,.\,\mathbf{0} + \beta\,.\,\mathbf{0}$$
$$= \alpha\,.\,O(x) + \beta\,.\,O(y)$$

$\Rightarrow\quad O : U \to V$ is a L.T. and consequently $O \in L(U, V)$.

For all $T_1 \in L(U, V)$ and $O \in L(U, V)$, we have

$$(O + T_1)(x) = O(x) + T_1(x) \quad \forall\ x \in U$$
$$= \mathbf{0} + T_1(x)$$

Thus $\quad (O + T_1)(x) = T_1(x) \qquad \forall\ x \in U$

Similarly $\ (T_1 + O)(x) = T_1(x) \qquad \forall\ x \in U$

Thus $\qquad O + T_1 = T_1 = T_1 + O.$

Hence O is the additive identity for L (U, V).

$$= (\alpha T_1)(x) + (\beta T_1)(x)$$
$$= (\alpha T_1 + \beta T_1)(x) \qquad \forall\ x \in U$$

Thus $\quad (\alpha + \beta) T_1 = \alpha T_1 + \beta T_1$.

V_8. For all $T_1 + T_2 \in L(U, V)$ and $\alpha \in F$, we have

$$(\alpha(T_1 + T_2))(x) = \alpha(T_1 + T_2)(x) \quad \forall x \in U$$
$$= \alpha(T_1(x) + T_2(x)) = \alpha T_1(x) + \alpha T_2(x) \qquad \text{[By distributivity in V]}$$
$$= (\alpha T_1 + \alpha T_2)(x)$$

Thus $\quad \alpha(T_1 + T_2) = \alpha T_1 + \alpha T_2$.

V_9. For all $T_1 \in L(U, V)$ and $\alpha, \beta \in F$, we have

$$((\alpha\beta) T_1)(x) = (\alpha\beta) T_1(x) \qquad \forall\ x \in U$$
$$= \alpha(\beta T_1(x)) \qquad \text{[By associative law]}$$
$$= \alpha(\beta T_1)(x)$$

Hence $\quad (\alpha\beta) T_1 = \alpha(\beta T_1)$.

V_{10}. For all $T_1 \in L(U, V)$, $\exists\ 1 \in F$, s.t.

$$(1\ T_1)(x) = 1(T_1(x)) \qquad \forall\ x \in U$$
$$= T_1(x)$$

$\Rightarrow \quad 1 . T = T_1$.

Thus all properties of vectors space are verified.

Hence L(U, V) is a vector space over the field F.

Theorem II. *Let U(F) be a finite dimensional vector space with $B = \{x_1, x_2, \ldots\ldots, x_n\}$ an ordered basis for U(F). Also V(F) be a vector space and $y_1, y_2, \ldots\ldots, y_n$ be any vector space in V(F). There exists a unique linear transformation T from U into V s.t.*

$$T(x_i) = y_i,\ \text{where}\ i = 1, 2, \ldots\ldots, n.$$

Proof. Existence of T.

$\forall\ x \in U$, $\exists$ unique scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ s.t.

$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n$

We define a mapping T as

$T(x) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots\ldots + \alpha_n y_n$

Clearly $T(x)$ is a unique element of V.

Thus T is a function U into V.

So each $x_i \in U$ can be represented as a linear combination of the vectors $\in$ B (basis), *i.e.*,

$$x_i = 0x_1 + 0x_2 + \ldots\ldots + 1 . x_i + \ldots\ldots + 0\, x_n$$

Thus $\quad T(x_i) = 0y_1 + 0y_2 + \ldots\ldots + 1 . y_i + \ldots\ldots + 0\, y_n$

i.e., $\quad T(x_i) = y_i,\ i = 1, 2, \ldots\ldots, n.$

(I) **To show** B_3 *is L.I.*

Let $\sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_{kl}T_{kl} = 0 \qquad \forall\, \alpha_{kl} \in F$

$\Rightarrow \left(\sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_{kl}T_{kl}\right)(x_i) = O\,(x_i)$ for $i = 1, 2, \ldots\ldots, n$

$\Rightarrow \sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_{kl}T_{kl}\ (x_i) = 0$ [*By linearity of* T_{kl} *and zero transformation* O]

$\Rightarrow \sum_{l=1}^{m}\alpha_{kl}T_{il}\ (x_i) = 0$ [$\because T_{kl}(x_i) = 0$ *when* $i \neq k$ *by* (*1*)]

$\Rightarrow \sum_{l=1}^{m}\alpha_{il}y_l = 0$

$\Rightarrow \alpha_{i1}y_1 + \alpha_{i2}y_2 + \ldots\ldots + \alpha_{im}y_m = 0$ for $i = 1, 2, \ldots\ldots, n$

$\Rightarrow \alpha_{i1} = \alpha_{i2} = \ldots\ldots = \alpha_{im} = 0$ for $i = 1, 2, \ldots\ldots, n$

Thus the set B_3 is L.I.

(II) **To show** L(U, V) = L (S).

Consider any linear transformation $T \in L$ (U, V) so that

$$T(x_i) \in U.$$

Thus $T(x_i)$ can be expressed as a linear combination of elements of B_2, being basis of U, and let

$$T(x_i) = \Sigma\beta_{il}y_l \qquad \ldots(3)$$

Consider $S = \sum_{k=1}^{n}\sum_{l=1}^{m}\beta_{kl}T_k$ ...(4)

Since S is a linear combination of elements of $B_3 \in L(U, V)$ and L(U, V) is a vector space so S is also a linear transformation $\in$ L(U, V).

Thus the result (2) will be true if S = T is proved.

From (4), $S(x_i) = \left(\sum_k\sum_l \beta_{kl}T_{kl}\right) x_i$ for $i = 1, 2, \ldots\ldots, n$

$= \sum_k\sum_l \beta_{ki}T_{kl}\ (x_i)$ [$\because$ T_{kl}*'s are linear*]

$= \sum_{l=1}^{m}\beta_{il}T_{il}\ (x_i)$ [$\because$ *From* (*1*), $T_k(x_i) = i$ *when* $k = i$]

$= \sum_{l=1}^{m}\beta_{il}y_l$ [$\because$ *of*(*1*)]

$= T(x_i)$ [$\because$ *of*(3)]

Again $\quad$ $TD\,(p(t)) = TD(p(t)) = T\left(\dfrac{d}{dt}(a_0 + a_1 t + a_2 t^2 + \ldots\ldots)\right)$

$$= T\,(a_1 + 2a_2 t + \ldots\ldots) = \int_0^t (a_1 + 2a_2 t + \ldots\ldots)\, dt$$

$$= \left[a_1 t + 2a_2 \frac{t^2}{2} + \ldots\ldots \right]_0^t = a_1 t + a_2 t^2 + \ldots\ldots$$

$\Rightarrow$ $\quad TD \neq p(t) \neq I(p(t)) \neq I$ $\quad$ ...(2)

(1) and (2) $\quad \Rightarrow DT \neq TD.$

THEOREMS

Theorem I. *Let U, V, W be vector spaces over the field F and*

$$T_1 : V \to W,\ T_2 : U \to V$$

be two linear transformations, then $T_1 T_2$ is a linear transformation from U to W. (G.N.D.U. 1987 S)

Proof. $T_1 : V \to W$ and $T_2 : U \to V$.

We define the mapping $T_1T_2 : U \to W$ as

$$(T_1T_2)(x) = T_1(T_2(x)) \quad \forall\ x \in V.$$

Let $x, y \in V$ and $\alpha, \beta \in F$, then

$T_1T_2\,(\alpha x + \beta y) = T_1\,(T_2(\alpha x + \beta y)) = T_1\,[\alpha T_2(x) + \beta T_2(y)]$

$$= \alpha T_1(T_2(x) + \beta T_1\ T_2(y)) = \alpha\,(T_1T_2)(x) + \beta\,(T_1T_2)(y)$$

Hence $T_1\ T_2$ is a linear transformation.

Theorem II. *Let U, V and W be three vector spaces over the field F. Let T_1, T_2 be linear transformations from U to V and S_1, S_2 be linear transformations from V to W, then*

(*i*) $S_1\,(T_1 + T_2) = S_1\,T_1 + S_1\,T_2$

(*ii*) $(S_1 + S_2)\,T_1 = S_1\,T_1 + S_2\,T_1$

(*iii*) $\alpha\,(S_1\,T_1) = (\alpha S_1)\,T_1 = S_1\,(\alpha T_1)$, *where* $\alpha \in F$.

Proof. Let x be any element $\in$ U.

(*i*) $\quad S_1(T_1 + T_2)(x) = S_1((T_1 + T_2)(x)) = S_1(T_1(x) + T_2(x))$

$$= S_1(T_1(x)) + S_1(T_2(x)) = (S_1T_1)(x) + (S_1T_2)(x)$$

$$= (S_1T_1 + S_1T_2)(x)$$

Hence $\quad S_1\,(T_1 + T_2) = S_1T_1 + S_1T_2$.

(*ii*) $\quad ((S_1 + S_2)\,T_1)(x) = (S_1 + S_2)\,(T_1(x)) = S_1\,(T_1(x)) + S_2\,(T_1(x))$

$$= (S_1T_1)(x) + (S_2T_1)(x) = (S_1T_1 + S_2T_1)(x)$$

Hence $\quad (S_1 + S_2)\,T_1 = S_1T_1 + S_2T_1$.

(*iii*) $\quad \alpha\,(S_1T_1)(x) = \alpha S_1(T_1(x)) = ((\alpha S_1)\,T_1)(x)$ $\quad$ ...(1)

and $\quad (S_1(\alpha T_1))(x) = S_1((\alpha T_1)(x)) = S_1\,(\alpha\,T_1(x))$

$$= \alpha S_1(T_1(x)) = ((\alpha S_1)\,T_1(x)) \quad \text{...(2)}$$

From (1) and (2),

$$\alpha\,(S_1T_1) = (\alpha S_1)\,T_1 = S_1\,(\alpha T_1).$$

Since the product of two linear transformations is again a linear transformation, therefore, any linear combination of linear transformations, as power of one linear transformation T, by the elements of F, is again a linear transformation. This is known as **polynomial linear transformation** in T over F.

In Symbols : $P(T) = a_0 I + a_1 T + a_2 T^2 + \ldots\ldots + a_n T^n$;

where a's $\in$ F.

The behaviour of such polynomials is the same as that of ordinary polynomials.

12. Inverses

Def. *A linear operator $T : V \to V$ is said to be* **invertible** *if there exists an operator $S : V \to V$ such that $TS = I = ST$.*

Here S is said to be the **inverse** of T and is written as T^{-1}.

THEOREMS

Theorem I. *Let T be a linear transformation on vector space V(F). Then T is invertible iff T is one-one and onto.* (Pbi.U. 1986 ; G.N.D.U. 1985)

Proof. Let $S : V \to V$ be defined as :

$\forall v \in V$ and T is onto, $\exists x \in V$ such that $T(x) = v$.

We take $S(v) = x$.

To prove. S is well defined.

Let $S(v) = x_1$ and $S(v) = x_2$

$\Rightarrow$ $T(x_1) = v$ and $T(x_2) = v$ [*By def.*]

$\Rightarrow$ $T(x_1) = T(x_2)$

$\Rightarrow$ $x_1 = x_2$.

Hence S is well defined.

Also $T : V \to V$ and $S : V \to V$

$\Rightarrow$ $TS : V \to V$ and $ST : V \to V$

$\therefore$ $\forall x \in V$, $(TS)(v) = T(S(v))$

$= T(x)$ [*By def.*]

$= v$. [*By def.*]

Thus $(TS)(v) = v = I(v)$ $\forall v \in V$

$\Rightarrow$ $TS = I$.

Similarly $ST = I$.

Thus $TS = I = ST$.

Hence T is invertible having S as its inverse.

Conversely. *T is one-one.*

Suppose that $T(v_1) = T(v_2)$ $\forall v_1 \neq v_2$

$\Rightarrow$ $S(T(v_1)) = S(T(v_2))$

$\Rightarrow$ $(ST)(v_1) = (ST)(v_2)$

$\Rightarrow$ $I(v_1) = (I)(v_2)$ [$\because$ $ST = I$]

$\Rightarrow$ $v_1 = v_2$, which leads to contradiction.

Hence T is one-one.

T is onto.

Now $S(v) = x \quad \forall x \in V$

$\Rightarrow \quad T(S(v)) = T(x)$

$\Rightarrow \quad (TS)(v) = T(x)$

$\Rightarrow \quad I(v) = T(x)$ $\quad [\because TS = I]$

$\Rightarrow \quad v = T(x)$

$\Rightarrow \quad T(x) = v \in V \quad \forall x \in V.$

Hence T is onto.

Theorem II. *Let T_1, T_2, T_3 be linear transformations on V such that*

$$T_1T_2 = T_3T_1 = I$$

Then T_1 is invertible and $T_1^{-1} = T_2 = T_3$. (P.U. 1985)

Proof. (*a*) **To prove.** T_1 *is invertible.*

For this, we have to prove

(*i*) T_1 is one-one (*ii*) T_1 is onto.

(*i*) *T_1 is one-one.* $T_1(x_1) = T_1(x_2)$

$\Rightarrow \quad T_3(T_1(x_1)) = T_3(T_1(x_2))$ $\quad [\because T_3$ *is a mapping*$]$

$\Rightarrow \quad (T_3T_1)(x_1) = (T_3T_1)(x_2)$

$\Rightarrow \quad I(x_1) = I(x_2)$ $\quad$ [*Given*]

$\Rightarrow \quad x_1 = x_2.$

Hence T_1 is one-one.

(*ii*) *T_1 is onto.*

$\forall\ y \in V, \exists\, x \in V$ s.t. $T_2(y) = x$ $\quad [\because T_2$ *is a mapping*$]$

Thus $\forall y \in V, \quad T_2(y) = x$

$\Rightarrow \quad T_1(T_2(y)) = T_1(x)$ $\quad [\because T_1$ *is a mapping*$]$

$\Rightarrow \quad (T_1T_2)(y) = T_1(x)$

$\Rightarrow \quad I(y) = T_1(x)$ $\quad$ [*Given*]

$\Rightarrow \quad y = T_1(x).$

Hence T_1 is onto.

Combining (*i*) and (*ii*), T_1 is invertible.

(*b*) **To prove.** $T_2^{-1} = T_2 = T_3$.

Now $T_1T_2 = I$

$\Rightarrow \quad T_1^{-1}(T_1T_2) = T_1^{-1}I$ $\quad [\because T_1$ *is invertible, proved in* (*a*)$]$

$\Rightarrow \quad (T_1^{-1}T_1)T_2 = T_1^{-1}$ $\quad$ [*By associativity*]

$\Rightarrow \quad IT_2 = T_1^{-1}$

$\Rightarrow \quad T_2 = T_1^{-1}$ $\quad$ ...(1)

Again $T_3T_1 = I$

$\Rightarrow$ $(T_3T_1)\, T_1^{-1} = IT_1^{-1}$ [$\because$ *T_1 is invertible, proved in (a)*]

$\Rightarrow$ $T_3\,(T_1T_1^{-1}) = T_1^{-1}$ [*By associativity*]

$\Rightarrow$ $T_3\, I = T_1^{-1}$

$\Rightarrow$ $T_3 = T_1^{-1}$ ...(2)

Combining (1) and (2), $T_1^{-1} = T_2 = T_3$.

Theorem III. Uniqueness of Inverse.

Let T be an invertible transformation on vector space V(F). Then the inverse of T is unique.

Proof. Let T_1 and T_2 be two inverses of T.

By def., $TT_1 = I = T_1T$ ...(1)

and $TT_2 = I = T_2T$ ...(2)

Now $T_2\,(TT_1) = T_2I$ [$\because$ *of (1)*]

$= T_2$ ...(3)

and $(T_2T)\, T_1 = IT_1$ [$\because$ *of (2)*]

$= T_1$ ...(4)

By associativity,

$$T_2\,(TT_1) = (T_2T)\, T_1,$$

$\therefore$ From (3) and (4),

$$T_2 = T_1.$$

Hence the inverse of T is unique.

Theorem IV. *Let T be a linear transformation on the vector space V(F) and T is invertible. Then the inverse mapping T^{-1} defined as*

$$y_0 = T(x_0) \Leftrightarrow T^{-1}(y_0) = x_0 \qquad \forall\ x_0, y_0 \in V$$

is a linear transformation.

Proof. Let $y_1, y_2 \in V$, $\exists$ unique elements $x_1, x_2 \in V$ s.t.

$T(x_1) = y_1$ and $T(x_2) = y_2$

$\Rightarrow$ $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$ [*By def.*]

For $x_1, x_2 \in V$ and $\alpha_1, \alpha_2 \in F$

$\Rightarrow$ $\alpha_1x_1 + \alpha_2x_2 \in V$.

Now $T(\alpha_1x_1 + \alpha_2x_2) = \alpha_1T(x_1) + \alpha_2T(x_2)$ [$\because$ *T is a linear*]

$= \alpha_1y_1 + \alpha_2y_2$

$\Rightarrow$ $T^{-1}(\alpha_1y_1 + \alpha_2y_2) = \alpha_1x_1 + \alpha_2x_2$

$= \alpha_1T^{-1}(y_1) + \alpha_2T^{-1}(y_2)$

Hence T^{-1} is a linear transformation.

Theorem V. *A linear transformation T on a finite dimensional vector space V(F) is invertible iff*

$$T(x) = \mathbf{0} \quad \Rightarrow \quad x = \mathbf{0}$$

Or iff T is non-singular

Or iff $\forall\ y \in V, \exists\, x$ *s.t.* $y = T(x)$ *i.e., T is onto.*

Proof. ($\Rightarrow$) T is invertible $\Rightarrow \begin{cases} (i) & T \text{ is one-one} \\ (ii) & T \text{ is onto} \end{cases}$

$$\Rightarrow \begin{cases} (i) & \mathrm{T}(x) = \mathbf{0} \Rightarrow \mathrm{T}(x) = \mathrm{T}(0) \Rightarrow x = \mathbf{0} \text{ for } x \in \mathrm{V} \\ (ii) & \text{for each } y \in \mathrm{V}, \exists\, x \text{ s.t. } y = \mathrm{T}(x). \end{cases}$$

($\Rightarrow$) (I) First of all, we shall show that if $\mathrm{T}(x) = \mathbf{0} \Rightarrow x = \mathbf{0}$, then T is invertible.

Now let $\mathrm{T}(x) = \mathbf{0} \Rightarrow x = \mathbf{0}$ for $x \in \mathrm{V}$ ...(1)

(*i*) **To prove.** *T is one-one.*

$\forall\, x_1, x_2 \in \mathrm{V}$,

$\mathrm{T}(x_1) = \mathrm{T}(x_2) \Rightarrow \mathrm{T}(x_1) - \mathrm{T}(x_2) = \mathbf{0}$

$\Rightarrow \mathrm{T}(x_1 - x_2) = \mathbf{0}$

$\Rightarrow x_1 - x_2 = \mathbf{0}$ [$\because$ *T is linear*]

$\Rightarrow x_1 = x_2$.

Hence T is one-one.

(*ii*) **To prove.** *T is onto.*

Let $\mathrm{B} = \{x_1, x_2, \ldots\ldots, x_n\}$ be one basis set of V.

To show. $\mathrm{B}_1 = \{\mathrm{T}(x_1), \mathrm{T}(x_2), \ldots\ldots, \mathrm{T}(x_n)\}$

is also a basis set of V.

Since T is a linear transformation from $\mathrm{V} \to \mathrm{V}$, therefore, $\mathrm{B}_1 \subset \mathrm{V}$, *i.e.*, B_1 is a subset on n dimensional vector space V(F). So in order to show that B_1 is a basis set of V, we have only to prove that B_1 is L.I.

To prove. *B_1 is L.I.*

Let $\sum_{i=1}^{n} \alpha_i \mathrm{T}(x_i) = \mathbf{0}\ \forall\ \alpha_i \in \mathrm{F}$

$\Rightarrow \mathrm{T}(\Sigma \alpha_i x_i) = \mathbf{0}$ [$\because$ *T is a linear*]

$\Rightarrow \Sigma \alpha_i x_i = \mathbf{0}$ [$\because$ *of (I)*]

$\Rightarrow$ each $\alpha_i = 0$ for $i = 1, 2, \ldots\ldots, n$ [$\because$ *B is L.I.*]

$\Rightarrow$ B_1 is L.I. and so B_1 is also basis set of V.

Now $\forall\ y \in \mathrm{V} \Rightarrow y = \sum_{i=1}^{n} \mathrm{B}_i \mathrm{T}(x_i)$ [$\because$ *B_i is basis of V*]

$\Rightarrow y = \mathrm{T}\left(\sum_{i=1}^{n} \mathrm{B}_i x_i\right)$ [$\because$ *T is a linear*]

$\Rightarrow y = \mathrm{T}(x)$ for $x \in \mathrm{V}$ [$\because$ *B is basis of V so $\Sigma B_i x_i = x \in V$*]

Thus T is onto.

Hence by (*i*) and (*ii*), T is invertible.

So $(\mathrm{T}(x) = \mathbf{0} \Rightarrow x = \mathbf{0}) \Rightarrow$ T is invertible.

(II) Secondly, we show that if T is onto $\Rightarrow$ T is invertible.

Now let T be onto, *i.e.*, $\forall\ y \in V\ \exists\ x \in V$ s.t. $y = T(x)$ ...(1)

Since T is onto, so in order to prove that T is invertible we have only to show that T is one-one.

(*i*) **To prove.** *T is one-one.*

Let $B_2 = \{y_1, y_2, \ldots\ldots, y_n\}$ be a basis of V.

Since T is onto, $\therefore\ \forall\ y_i \in B_2\ \exists\ x_i \in V$ s.t.

$$y_i = T(x_i) \qquad \ldots(2)$$

Now we show that $B = \{x_1, x_2, \ldots\ldots, x_n\}$ is a basis of V under the condition (2).

Again in order to show that B as basis of V, we have to prove that B is L.I.

To prove. *B is L.I.*

Let $\sum_{i=1}^{n} \alpha_i x_i = 0 \qquad \forall\ \alpha_i \in F$

$\Rightarrow \quad T(\Sigma\alpha_i x_i) = T(0)$ [$\because$ *T is a mapping*]

$\Rightarrow \quad \Sigma\alpha_i\, T(x_i) = 0$ [$\because$ *T is a linear*]

$\Rightarrow \quad \Sigma\alpha_i y_i = 0$ [*By* (2)] [$\because$ *B is L.I.*]

$\Rightarrow \quad$ each $\alpha_i = 0$ for $i = 1, 2, \ldots\ldots, n.$

Thus B is L.I. So B is also a basis of V.

Now if $x \in V \Rightarrow x = \Sigma\, B_i x_i$ For B_i's $\in V$ ...(3)

So $T(x) = 0 \Rightarrow T(\Sigma\, B_i x_i) = 0$ [$\because$ *B is basis of V*]

$$\Rightarrow \sum_{i=1}^{n} B_i T(x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \beta_i y_i = 0$$

$\Rightarrow$ each $B_i = 0$ [$\because$ *B is L.I.*]

$\Rightarrow x = 0$ [$\because$ *of* (3)]

Thus T is one-one.

Also T is onto [*Given*]

$\therefore$ T is invertible.

Hence T is onto $\Rightarrow$ T is invertible.

Theorem VI. Algebra of Invertibles.

Let V(F) be a vector space and T_1, T_2 be linear transformations on V. Then (i) if T_1 and T_2 are invertible, then $T_1 T_2$ is invertible

and $\qquad (T_1T_2)^{-1} = T_2^{-1}T_1^{-1}$. (G.N.D.U. 1990, 85 ; Pbi.U. 1986)

(*ii*) *If T_1 is invertible and $\alpha \neq 0 \in F$, then αT_1 is invertible and*

$$(\alpha T_1)^{-1} = 1/\alpha T_1^{-1}. \qquad \text{(G.N.D.U. 1985)}$$

(*iii*) *If T_1 is invertible, then T_1^{-1} is invertible, and $(T_1^{-1})^{-1} = T_1$.* (Pbi.U. 1986)

Proof. *Given.* T_1 and T_2 are invertible,

$\therefore \quad T_1T_1^{-1} = T_1^{-1}T_1 = I$ ...(1)

and $\quad T_2T_2^{-1} = T_2^{-1}T_2 = I$ ...(2)

Also T_1^{-1}, T_2^{-1} are linear transformations.

(*i*) **To prove.** *T_1T_2 is one-one.*

$\forall\ x_1, x_2 \in V$ if

$$(T_1T_2)(x_1) = (T_1T_2)(x_2)$$

$\Rightarrow \quad T_1(T_2(x_1)) = T_1(T_2(x_2))$

$\Rightarrow \quad T_2(x_1) = T_2(x_2)$ [$\because$ *T_1 is one-one*]

$\Rightarrow \quad x_1 = x_2$

Hence T_1T_2 is one-one. ...(3)

To prove. *T_1T_2 is onto.*

$\forall\ y \in V\ \exists\ x \in V$ s.t. $T_1(x) = y$ [$\because$ *T_1 is onto*]

and $\quad \forall\ x \in V\ \exists\ z \in V$.s.t. $T_2(z) = x$ [$\because$ *T_2 is onto*]

$\therefore \quad \forall\ y \in V\ \exists\ z \in V$ s.t. $y = T_1(x) = T_1(T_2(z)) = (T_1T_2)(z)$

Hence T_1T_2 is onto. ...(4)

(3) and (4) $\Rightarrow$ T_1T_2 is invertible.

Further since T_1^{-1} and T_2^{-1} are linear transformations.

$\therefore \quad (T_1T_2)(T_2^{-1}T_1^{-1}) = T_1(T_2T_2^{-1})T_1^{-1}$

$= T_1(I)T_1^{-1}$ [$\because$ *of* (2)]

$= T_1T_1^{-1} = I$ ...(A) [$\because$ *of* (1)]

Similarly, $(T_2^{-1}T_1^{-1})(T_1T_2) = I$ ...(B)

(A) and (B) $\Rightarrow$ $(T_1T_2)(T_2^{-1}T_1^{-1}) = (T_2^{-1}T_1^{-1})(T_1T_2) = I$

Hence $\quad (T_1T_2)^{-1} = T_2^{-1}T_1^{-1}$. [$\because$ *T_1T_2 is invertible*]

(*ii*) **To prove.** *αT_1 is one-one.*

$\forall\ x_1, x_2 \in V$, if

$$(\alpha T_1)(x_1) = (\alpha T_1)(x_2)$$

$\Rightarrow \quad \alpha T_1(x_1) = \alpha T_1(x_2)$

$\Rightarrow \quad T_1(x_1) = T_1(x_2)$ [$\because \alpha \neq 0$]

$\Rightarrow \quad x_1 = x_2$ [$\because$ *T_1 is one-one*]

Hence αT_1 is one-one.

Since $\quad (\alpha T_1)(x) = \alpha T_1(x) \quad \forall\ x \in V$

and T_1 is onto.

So αT_1 is also onto.

Since αT_1 one-one onto, therefore, αT_1 is invertible and $(\alpha T_1)(\alpha^{-1} T_1^{-1}) = \alpha\alpha^{-1} T_1 T_1^{-1}$ [*As* $\alpha \neq 0$ *so* α^{-1} *exists*]

$= I \, . \, I$

$= I.$

Hence $(\alpha T_1)^{-1} = \alpha^{-1} T_1^{-1} = \dfrac{1}{\alpha} T_1^{-1}.$ [$\because \alpha T_1$ *is invertible*]

(*iii*) **To prove.** T_1^{-1} *is one-one.*

Let $y_1 = T_1(x_1)$ for $y_1, x_1 \in V \Rightarrow T_1^{-1}(y_1) = x_1$ ...(1)

[$\because T_1$ *is invertible*]

and $y_2 = T_1(x_2)$ for $y_2, x_2 \in V \Rightarrow T_1^{-1}(y_2) = x_2$ ...(2)

$\forall\ y_1, y_2 \in V$, if

$T_1^{-1}(y_1) = T_1^{-1}(y_2)$

$\Rightarrow \quad x_1 = x_2$ [$\because$ *of (1) and (2)*]

$\Rightarrow \quad T_1(x_1) = T_1(x_2)$ [$\because T_1$ *is a mapping*]

$\Rightarrow \quad y_1 = y_2$ [$\because$ *of (1) and (2)*].

Hence T_1^{-1} is one-one.

To prove. T_1^{-1} *is onto.*

Since T_1 is a mapping,

$\therefore \quad \forall \quad x \in V\ \exists\ y \in V$ s.t. $y = T_1(x)$

$\Rightarrow \quad \forall \quad x \in V\ \exists\ y \in V$ s.t. $T_1^{-1}(y) = x$ [$\because T_1$ *is invertible so* $y = T_1(x) \Rightarrow T_1^{-1}(y) = x$]

Hence T_1^{-1} is onto.

Since T_1^{-1} is one-one onto, so T_1^{-1} is invertible.

Also by (1), $(T_1 T_1^{-1}) = (T_1^{-1} T_1) = I$

Hence $(T_1^{-1})^{-1} = T_1.$ [$\because T_1^{-1}$ *is invertible*]

SOLVED EXAMPLES

Example 1. *Fill up the blanks in the following statements :*

(*i*) *A linear operator T on* R^2 *defined by* $T(x, y) = (ax + by, cx + dy)$ *will be invertible iff......*

(*ii*) *If T is a linear operator on* R^2 *defined by* $T(x, y) = (x - y, y)$ *then* $T^2(x, y) = ...$

Sol. (*i*) $ad - bc \neq 0$, (*ii*) $(x - 2y, y)$.

Example 2. *State whether the following statements are true or false :*

(*i*) *For two linear operators* T_1 *and* T_2 *on* R^2

$T_1 T_2 = O \Rightarrow T_2 T_1 = O.$

(*ii*) *If S and T be linear operators on a vector space V, then*

$(S + T)^2 = S^2 + 2ST + T^2.$

Sol. (*i*) False, (*ii*) False.

Example 3. *Prove that there is no non-singular linear transformation from* R^4 *to* R^3.

Sol. If there is a non-singular linear transformation T,

$\therefore$ dim. (Range T) = dim. R^4 [$\because$ *dim. (Range T) = dim. V*]

$\Rightarrow$ dim. (Range T) = 4, which is impossible because Range T is a subset R^3, and dim. $R^3 = 3$.

Hence dim. (Range T) can't exceed 3 (= dim. R^3).

Example 4. *If $T_1 : V \to W$ and $T_2 : U \to V$ are two linear transformations, prove that T_1T_2 is also a linear transformation.*

Sol. Since range of T_2 is in the domain of T_1,

$\therefore$ T_1T_2 is defined and

$$(T_1T_2)(x) \text{ for } x \in U = T_1(T_2(x)) = T_1(y) \in W,$$

where $y = T_2(x) \in V \ \forall \ x \in U$

i.e., $T_1T_2 : V \to W$.

$$\forall \ x, y \in U \text{ and } \alpha, \beta \in F \ \Rightarrow \ \alpha x + \beta y \in U$$

$$\therefore \ (T_1T_2)(\alpha x + \beta y) = T_1[T_2(\alpha x + \beta y)] = T_1[\alpha T_2(x) + \beta T_2(y)] \qquad [\text{By linearity of } T_2]$$

$$= \alpha[T_1(T_2(x))] + \beta[T_1(T_2(y))] \qquad [\text{By linearity of } T_1]$$

$$= \alpha(T_1T_2)(x) + \beta(T_1T_2)(y)$$

Thus T_1T_2 is linear.

Hence T_1T_2 is a linear transformation from U into W.

Example 5. *Show that $T : R^3 \to R^3$ defined by*

$$T(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$$

is non-singular, where θ is any angle. (G.N.D.U. 1995 S)

Sol. Clearly if $x = 0$, $y = 0$, $z = 0$, then the image of (0, 0, 0) under T is (0, 0, 0).

But if $(x, y, z) \neq (0, 0, 0)$, then the image is not zero for any θ.

Thus (0, 0, 0) is the only vector in the null space of T.

Hence T is non-singular.

Example 6. (*a*) *Let T_1 and T_2 be linear operators on R^2 defined by*

$$T_1(x, y) = (0, x) \text{ and } T_2(x, y) = (y, x).$$

Compute $T_2 + T_1$, $2T_2 - 3T_1$, T_2T_1, T_1T_2, T_2^2, T_1^2.

(*b*) *Let T_1 and T_2 be linear operators on R^2 defined by*

$$T_1(x, y) = (y, x) \text{ and } T_2(x, y) = (x, 0).$$

Compute $T_2 + T_1$, T_2T_1, T_1T_2, T_1^2, T_2^2.

Sol. (*a*) Since L (R^2, R^2) is a vector space, which is closed both for addition and scalar multiplication.

$\therefore$ $T_2 + T_1$, $2T_2 - 3T_1$, T_2T_1, T_1T_2, T_2^2, T_1^2 will be all linear transformations.

We have $T_1(x, y) = (0, x)$ and $T_2(x, y) = (y, x)$.

(*i*) $$(T_2 + T_1)(x, y) = T_2(x, y) + T_1(x, y) = (y, x) + (0, x)$$
$$= (y + 0, x + x) = (y, 2x).$$

(*ii*) $$(2T_2 - 3T_1)(x, y) = (2T_2)(x, y) - (3T_1)(x, y) = 2T_2(x, y) - 3T_1(x, y)$$
$$= 2(y, x) - 3(0, x) = (2y, 2x) + (0, -3x)$$
$$= (2y + 0, 2x - 3x) = (2y, -x).$$

(*iii*) $$(T_2T_1)(x, y) = T_2[T_1(x, y)] = T_2(0, x) = (x, 0).$$

(*iv*) $$(T_1T_2)(x, y) = T_1[T_2(x, y)] = T_1(y, x) = (0, y).$$

(v) $(T_2^2)(x, y) = T_2[T_2(x, y)] = T_2(y, x) = (x, y)$.

(vi) $(T_1^2)(x, y) = T_1[T_1(x, y)] = T_1(0, x) = (0, 0)$.

(b) As in part (a), we have

$$T_1(x, y) = (y, x) \text{ and } T_2(x, y) = (x, 0).$$

(i) $(T_2 + T_1)(x, y) = T_2(x, y) + T_1(x, y) = (x, 0) + (y, x)$

$= (x + y, 0 + x) = (x + y, x)$.

(ii) $(T_2T_1)(x, y) = T_2[T_1(x, y)] = T_2(y, x) = (y, 0)$.

(iii) $(T_1T_2)(x, y) = T_1[T_2(x, y)] = T_1(x, 0) = (0, x)$.

(iv) $(T_1^2)(x, y) = T_1[T_1(x, y)] = T_1(y, x) = (x, y)$.

(v) $(T_2^2)(x, y) = T_2[T_2(x, y)] = T_2(x, 0) = (x, 0)$.

Example 7. (a) *Let $T_1 : R^3 \to R^2$ and $T_2 : R^3 \to R^2$ be defined by*

$$T_1(a, b, c) = (3a, b + c) \text{ and } T_2(a, b, c) = (2a - 3c, b).$$

Compute $T_1 + T_2$, $5T_1$, $4T_1 - 5T_2$, T_1T_2 and T_2T_1.

(b) *Let $T_1 : R^3 \to R^2$ and $T_2 : R^3 \to R^2$ be defined by*

$$T_1(x, y, z) = (2x, y + z) \text{ and } T_2(x, y, z) = (x - z, y).$$

Compute $T_1 + T_2$, $3T_1$, $2T_1 - 5T_2$.

(c) *Let $T_1 : R^3 \to R^2$ and $T_2 : R^3 \to R^2$ be defined by*

$$T_1(x, y, z) = (5x, 2y + z) \text{ and } T_2(x, y, z) = (x - z, y).$$

Compute $3T_1 - 4T_2$.

Sol. (a) Since L (R^3, R^2) is a vector space, which is closed both for addition and scalar multiplication.

$\therefore$ $T_1 + T_2$, $5T_1$ and $4T_1 - 5T_2$ will be all linear transformations.

(i) $T_1 + T_2 : R^3 \to R^2$. Let $\alpha = (a, b, c) \in R^3$

$\therefore$ $(T_1 + T_2)\alpha = (T_1)\alpha + (T_2)\alpha = T_1(\alpha) + T_2(\alpha)$

$= T_1(a, b, c) + T_2(a, b, c) = (3a, b + c) + (2a - 3c, b)$ [*By def.*]

$= (3a + 2a - 3c, b + c + b) = (5a - 3c, 2b + c)$.

(ii) $5T_1 : R^3 \to R^2$. Let $\alpha = (a, b, c) \in R^3$

$\therefore$ $(5T_1)\alpha = 5T_1(\alpha) = 5T_1(a, b, c) = 5(3a, b + c)$ [*By def.*]

$= (15a, 5b + 5c)$.

(iii) $4T_1 - 5T_2 : R^3 \to R^2$. Let $\alpha = (a, b, c) \in R^3$

$\therefore$ $(4T_1 - 5T_2)\alpha = (4T_1)\alpha - (5T_1)\alpha = 4T_1(\alpha) - 5T_2(\alpha)$

$= 4T_1(a, b, c) - 5T_2(a, b, c) = 4(3a, b + c) - 5(2a - 3c, b)$ [*By def.*]

$= (12a, 4b + 4c) - (10a - 15c, 5b) = (12a - 10a + 15c, 4b + 4c - 5b)$

$= (2a + 15c, -b + 4c)$.

(iv)–(v) Both T_1T_2 and T_2T_1 are not defined because in each case the range of post-factor transformation is not equal to the domain of pre-factor transformation.

(b) Since L (R^3, R^2) is a vector space which is closed both for addition and scalar multiplication,

$\therefore$ $T_1 + T_2$, $3T_1$ and $2T_1 - 5T_2$ will be all linear transformations.

(*i*) $T_1 + T_2 : R^3 \to R^2$. Let $\alpha = (x, y, z) \in R^3$

$\therefore \quad (T_1 + T_2)\alpha = (T_1)\alpha + (T_2)\alpha = T_1(\alpha) + T_2(\alpha)$

$= T_1(x, y, z) + T_2(x, y, z) = (2x, y + z) + (x - z, y)$ [*By def.*]

$= (2x + x - z, y + z + y) = (3x - z, 2y + z).$

(*ii*) $3T_1 : R^3 \to R^2$. Let $\alpha = (x, y, z) \in R^3$

$\therefore \quad (3T_1)\alpha = 3T_1(x, y, z) = 3\,(2x, y + z)$ [*By def.*]

$= (6x, 3y + 3z).$

(*iii*) $2T_1 - 5T_2 : R^3 \to R^2$. Let $\alpha = (x, y, z) \in R^3$

$\therefore \quad (2T_1 - 5T_2)\alpha = (2T_1)\alpha - (5T_2)\alpha = 2T_1(\alpha) - 5T_2(\alpha)$

$= 2T_1(x, y, z) - 5T_2(x, y, z) = 2\,(2x, y + z) - 5\,(x - z, y)$ [*By def.*]

$= (4x, 2y + 2z) - (5x - 5z, 5y) = (4x - 5x + 5z, 2y + 2z - 5y)$

$= (-x + 5z, -3y + 2z).$

(*c*) Since L (R^3, R^2) is a vector space which is closed both for addition and scalar multiplication.

$\therefore$ $3T_1 - 4T_2$ will be a linear transformation.

$3T_1 - 4T_2 : R^3 \to R^2$. Let $\alpha = (x, y, z) \in R^3$

$\therefore \quad (3T_1 - 4T_2)\alpha = (3T_1)\alpha - (4T_2)\alpha = 3T_1(\alpha) - 4T_2(\alpha)$

$= 3T_1(x, y, z) - 4T_2(x, y, z) = 3\,(5x, 2y + z) - 4\,(x - z, y)$

$= (15x, 6y + 3z) - (4x - 4z, 4y) = (15x - 4x + 4z, 6y + 3z - 4y)$

$= (11x + 4z, 2y + 3z).$

Example 8. *Let* $T_1 : R^3 \to R^2, T_2 : R^3 \to R^2$ *and* $T_3 : R^2 \to R^2$ *be defined as*

$T_1(x, y, z) = (y, x + z)$

$T_2(x, y, z) = (2z, x - y),\ T_3(x, y) = (y, 2x).$

Find formulae defining the mappings

(*i*) T_3T_1 *and* T_3T_2 (*ii*) T_1T_3 *and* T_2T_3

(*iii*) $T_3(T_1 + T_2)$ *and* $T_3T_1 + T_3T_2$.

Sol. (*i*) $(T_3T_1)(x, y, z) = T_3\,[T_1(x, y, z)] = T_3(y, x + z) = (x + z, 2y)$

and $(T_3T_2)(x, y, z) = T_3\,[T_2(x, y, z)] = T_3\,(2z, x - y)$

$= (x - y, 4z).$

(*ii*) $(T_1T_3)(x, y, z) = T_1\,[T_3(x, y, z)]$

Here $T_3\,(x, y, z)$ is not defined.

Hence T_1T_3 is not defined.

Similarly T_2T_3 is also not defined.

(*iii*) $T_3\,(T_1 + T_2)(x, y, z) = T_3\,[(T_1 + T_2)(x, y, z)] = T_3\,[T_1(x, y, z) + T_2(x, y, z)]$

$= T_3\,[(y, x + z) + (2z, x - y)] = T_3\,(y + 2z, 2x + z - y)$

$= (2x + z - y, 2y + 4z).$

And $(T_3T_1 + T_3T_2)(x, y, z) = (T_3T_1)(x, y, z) + (T_3T_2)(x, y, z)$

$= T_3\,[T_1(x, y, z)] + T_3\,[T_2(x, y, z)]$

$= T_3\,(y, x + z) + T_3\,(2z, x - y) = (x + z, 2y) + (x - y, 4z)$

$= (2x - y + z, 2y + 4z).$

Example 9. *(a) Let $T_1 : R^3 \to R^2$ s.t. $T_1(a, b, c) = (3a, 4b - c)$*

$$T_2 : R^2 \to R^2 \text{ s.t. } T_2\ (a, b) = (-a, b).$$

Compute T_1T_2 and T_2T_1.

(b) Let $T_1 : R^3 \to R^2$ and $T_2 : R^2 \to R^2$ s.t.

$$T_1\ (a, b, c) = (2a, b + c) \text{ and } T_2\ (a, b) = (b, a).$$

Compute T_1T_2 and T_2T_1.

Sol. (*a*) Since the range of T_2, *i.e.*, R^2 is not contained in the domain of T_1, *i.e.*, R^3.

$\therefore$ T_1T_2 is not defined.

But the range of T_1, *i.e.*, R^2 is equal to domain of T_2, *i.e.*, R^2,

$\therefore$ T_2T_1 is defined and in this case T_2T_1 is a transformation from R^3 to R^2.

Let $\qquad x = (a, b, c) \in R^3$

$$\therefore \quad (T_2T_1)\ (x) = T_2\ (T_1\ (x)) = T_2\ (T_1(a, b, c))$$
$$= T_2(3a, 4b - c) = (-3a, 4b - c).$$

(*b*) Since the range of T_2, *i.e.*, R^2 is not contained in the domain of T_1, *i.e.*, R^3,

$\therefore$ T_1T_2 is not defined.

But range of T_1, *i.e.*, R^2 is equal to domain of T_2, *i.e.*, R^2,

$\therefore$ T_2T_1 is defined and in this case T_2T_1 is a transformation from R^3 to R^2.

Let $\qquad x = (a, b, c) \in R^3$

$$\therefore \quad (T_2\ T_1)\ (x) = T_2\ (T_1\ (x)) = T_2\ (T_1\ (a, b, c))$$
$$= T_2\ (2a, b + c) = (b + c, 2a).$$

Example 10. *Let $T : R^3 \to R^2$ and $S : R^2 \to R^3$ be linear transformations defined by $T\ (x, y, z) = (x - 3y - 2z, y - 4z)$, $S\ (x, y) = (2x, 4x - y, 2x + 3y)$. Find ST and TS. Is product commutative?* (P.U. 1989)

Sol. (*i*) ST is defined because range of T = domain of S = 2

$$\therefore \quad ST\ (x, y, z) = S\ [T\ (x, y, z)] = S\ (x - 3y - 2z, y - 4z)$$
$$= (2x - 6y - 4z, 4(x - 3y - 2z) - (y - 4z), 2\ (x - 3y - 2z) + 3\ (y - 4z))$$
$$= (2x - 6y - 4z, 4x - 13y - 4z, 2x - 3y - 16z) \qquad ...(1)$$

(*ii*) TS is defined because range of S = domain of T = 3

$$\therefore \quad TS\ (x, y) = T\ [S\ (x, y)]$$
$$= T\ (2x, 4x - y, 2x + 3y)$$
$$= (2x - 3(4x - y) - 2\ (2x + 3y), 4x - y - 4\ (2x + 3y))$$
$$= (-14x - 3y, -4x - 13y) \qquad ...(2)$$

From (1) and (2), ST $\neq$ TS

i.e., product is not commutative.

Example 11. *Let $T : R^3 \to R^2$ and $S : R^2 \to R^2$ be defined as follows:*

$$T\ (x, y, z) = (2x + y, -3y + z), S\ (x, y) = (3x - y, -x + 3y).$$

Find a formula for ST. Does TS exist? (G.N.D.U. 1992)

Sol. ST is defined because range of T = domain of S = 2.

$$\therefore \quad ST(x, y, z) = S[T(x, y, z)] = S(2x + y, -3y + z)$$
$$= (3(2x + y) - (-3y + z), -(2x + y) + 3(-3y + z))$$
$$= (6x + 3y + 3y - z, \ -2x - y - 9y + 3z)$$
$$= (6x + 6y - z, -2x - 10y + 3z).$$

TS is not defined because range of S $\neq$ domain of T $[\because 2 \neq 3]$

Hence TS does not exist.

Example 12. *Let $T_1, T_2 : R^2 \to R^2$ be defined as*

$$T_1(a, b) = (a, 0), T_2(a, b) = (0, a).$$

Prove that $T_1T_2 = O$, $T_2T_1 \neq O$, $T_1^2 = T_1$.

Sol. We have $T_2T_1 : R^2 \to R^2$ $[\because T_1, T_2 : R^2 \to R^2]$

(*i*) $(T_1T_2)(a, b) = T_1(T_2(a, b)) = T_1(0, a) = (0, 0)$.

Hence $T_2T_1 = O$.

(*ii*) $(T_2T_1)(a, b) = T_2(T_1(a, b)) = T_2(a, 0) = (0, a) \neq (0, 0)$.

Hence $T_2T_1 \neq O$.

(*iii*) $T_1^2(a, b) = T_1(T_1(a, b)) = T_1(a, 0) = (a, 0) = T_1(a, b)$

Hence $T_1^2 = T_1$.

Exmaple 13. *Find two sets of linear transformations T_1 and $T_2 : R^2 \to R^2$ such that $T_2T_1 = O$ and $T_1T_2 \neq O$.*

Sol. One of the sets of linear transformations is defined in Ex. 12.

The other set can be defined as

$$T_1(a, b) = (0, 6a) \text{ and } T_2(a, b) = (2a, 0).$$

Here again as in Ex. 12, $T_2T_1 = O$ while $T_1T_2 \neq O$.

Example 14. *Show that the linear operator T on R^3 is invertible, and find a formula for T^{-1}, where $T(x, y, z) = (x - 3y - 2z, x - 4z, z)$.* (G.N.D.U. 1992 S)

Sol. We know that T is invertible if T is one-one and onto.

Let $r = (x_1, y_1, z_1)$, $s = (x_2, y_2, z_2) \in R^3$,

then $T(r) = T(s)$

$\Rightarrow \quad (x_1 - 3y_1 - 2z_1, x_1 - 4z_1, z_1) = (x_2 - 3y_2 - 2z_2, x_2 - 4z_2, z_2)$

$\Rightarrow \quad x_1 - 3y_1 - 2z_1 = x_2 - 3y_2 - 2z_2, x_1 - 4z_1 = x_2 - 4z_2, z_1 = z_2$

$\Rightarrow \quad x_1 = x_2, y_1 = y_2, z_1 = z_2$

$\Rightarrow \quad r = s.$

Thus T is one-one.

Let $(a, b, c) \in R^3$. We shall show that $\exists$ a vector in R^3 whose T image is (a, b, c).

Let that vector be (x, y, z) so that

$$T(x, y, z) = (a, b, c) = (x - 3y - 2z, x - 4z, z)$$

$\Rightarrow \quad a = x - 3y - 2z, \ b = x - 4z, \ c = z$

$\Rightarrow \quad x = b + 4c, \ y = \dfrac{1}{3}(-a + b + 2c), z = c.$

Since $a, b, c \in R$, $\therefore$ x, y, z also $\in R$ so that $(x, y, z) \in R^3$.

Thus T is onto.

Hence T is invertible

$$T(x, y, z) = (a, b, c)$$

$$\Rightarrow \quad T^{-1}(a, b, c) = (x, y, z)$$

Hence $T^{-1}(a, b, c) = \left(b + 4c, \frac{-a}{3} + \frac{b}{3} + \frac{2}{3}c, c\right)$.

Example 15. *Let $T : V_3(R) \to V_3(R)$ be defined as*

$$T(a, b, c) = (3a, a - b, 2a + b + c).$$

Prove that T is invertible and find T^{-1}.

Also prove that $(T^2 - I)(T - 3I) = O$. (P.U. 1996 ; G.N.D.U. 1986, 85 S ; Pbi. U. 1985)

Sol. (*i*) We know that T is invertible if T is one-one and onto.

Let $x_1 = (a_1, b_1, c_1), x_2 = (a_2, b_2, c_2) \in V_3(R)$,

then $\quad T(x_1) = T(x_2) \Rightarrow (3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$

$$\Rightarrow \quad 3a_1 = 3a_2, a_1 - b_1 = a_2 - b_2, 2a_1 + b_1 + c_1 = 2a_2 + b_2 + c_2$$

$$\Rightarrow \quad a_1 = a_2, b_1 = b_2, \ c_1 = c_2$$

$$\Rightarrow \quad x_1 = x_2.$$

Thus T is one-one.

Let $(r, s, t) \in V_3(R)$. We shall show that $\exists$ a vector in $V_3(R)$ whose T image is (r, s, t).

Let that vector be (a, b, c) so that

$$T(a, b, c) = (r, s, t) = (3a, a - b, 2a + b + c)$$

$$\Rightarrow \quad r = 3a, \ s = a - b, \ t = 2a + b + c$$

$$\Rightarrow \quad a = \frac{r}{3}, \ b = \frac{r}{3} - s, c = t - r + s.$$

Since $r, s, t \in R$, $\quad \therefore$ a, b, c also $\in R$ so that

$(a, b, c) \in V_3(R)$.

Thus T is onto.

Hence T is invertible.

$$T(a, b, c) = (r, s, t)$$

$$\Rightarrow \quad T^{-1}(r, s, t) = (a, b, c)$$

Hence $\quad T^{-1}(r, s, t) = \left(\frac{r}{3}, \frac{r}{3} - s, t - r + s\right)$.

(*ii*) $\quad (T - 3I)(a, b, c) = T(a, b, c) - 3I(a, b, c) = (3a, a - b, 2a + b + c) - 3I(a, b, c)$

$$= (3a, a - b, 2a + b + c) + (-3a, -3b, -3c)$$

$$= (3a - 3a, a - b - 3b, 2a + b + c - 3c) = (0, a - 4b, 2a + b - 2c).$$

$\therefore$ $(T^2 - I)(T - 3I)(a, b, c) = (T^2 - I)[(T - 3I)(a, b, c)]$

$$= (T^2 - I)(0, a - 4b, 2a + b - 2c)$$

$$= T^2(0, a - 4b, 2a + b - 2c) - I(0, a - 4b, 2a + b - 2c)$$

$$= T^2(A, B, C) - I(A, B, C) \qquad \ldots(1),$$

(e) $T(x, y, z) = (x - 2y - z, y - z, x)$. (P.U. 1993 S, 85 S)

Sol. (a) (i) Let W be the null space of T.

So W is the set of all (x, y, z) such that

$$T(x, y, z) = (0, 0, 0)$$

i.e., $$(2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$$

$\Rightarrow$ W is the solution space of

$$2x = 0,\ 4x - y = 0,\ 2x + 3y - z = 0,$$

which has (0, 0, 0) as a trivial solution.

Thus $W = \{0\}$.

Hence T is non-singular and hence is invertible.

(ii) Let $T(x, y, z) = (r, s, t)$

$\therefore$ $T^{-1}(r, s, t) = (x, y, z)$.

Now $T(x, y, z) = (2x, 4x - y, 2x + 3y - z) = (r, s, t)$

$\Rightarrow$ $2x = r,\ 4x - y = s,\ 2x + 3y - z = t$

$\Rightarrow$ $x = \frac{1}{2} r,\ y = 2r - s,\ z = 7r - 3s - t.$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = \left(\frac{1}{2}r, 2r - s, 7r - 3s - t\right).$$

(b) (i) Let W be the null space of T.

So W is the set of all (x, y, z) such that

$$T(x, y, z) = (0, 0, 0)$$

i.e., $(x - 3y - 2z, y - 4z, z) = (0, 0, 0)$

$\Rightarrow$ W is the solution space of

$$x - 3y - 2z = 0,\ y - 4z = 0,\ z = 0,$$

which has (0, 0, 0) as trivial solution.

Thus $W = \{0\}$.

Hence T is non-singular and hence is invertible.

(ii) Let $(x, y, z) = (r, s, t)$

$\therefore$ $T^{-1}(r, s, t) = (x, y, z)$.

Now $T(x, y, z) = (x - 3y - 2z, y - 4z, z) = (r, s, t)$

$\Rightarrow$ $x - 3y - 2z = r,\ y - 4z = s,\ z = t$

$\Rightarrow$ $x = 14t + 3s + r,\ y = 4t + s,\ z = t.$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = (14t + 3s + r, 4t + s, t).$$

(c) (i) Let W be the null space of T.

So W is the set of all (x, y, z) such that $T(x, y, z) = (0, 0, 0)$

i.e., $(x + z,\ x - z,\ y) = (0, 0, 0)$

$\Rightarrow$ W is the solution space of

$$x + z = 0,\ x - z = 0,\ y = 0,$$

which has $(0, 0, 0)$ as trivial solution.

Thus $W = \{0\}$.

Hence T is non-singular and hence is invertible.

(*ii*) Let $(x, y, z) = (r, s, t)$

$\therefore$ $T^{-1}(r, s, t) = (x, y, z)$.

Now $T(x, y, z) = (x + z, x - z, y) = (r, s, t)$

$\Rightarrow$ $x + z = r,\ x - z = s,\ y = t$

$\Rightarrow$ $x = \frac{1}{2}(r + s),\ z = \frac{1}{2}(r - s),\ y = t.$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = \left(\frac{1}{2}r + \frac{1}{2}s, t, \frac{1}{2}r - \frac{1}{2}s\right).$$

(*d*) (*i*) Let W be the null space of T.

So W is the set of all (x, y, z) such that

$$T(x, y, z) = (0, 0, 0)$$

i.e., $(3x, x - y, 2x + y + z) = (0, 0, 0),$

$\Rightarrow$ W is the solution space of

$$3x = 0,\ x - y = 0,\ 2x + y + z = 0,$$

which has $(0, 0, 0)$ as trivial solution.

Thus $W = \{0\}$.

Hence T is non-singular and hence is invertible.

(*ii*) Let $T(x, y, z) = (r, s, t)$

$\therefore$ $T^{-1}(r, s, t) = (x, y, z)$.

Now $T(x, y, z) = (3x, x - y, 2x + y + z) = (r, s, t)$

$\Rightarrow$ $3x = r,\ x - y = s,\ 2x + y + z = t$

$\Rightarrow$ $x = \frac{1}{3}r,\ y = \frac{1}{3}r - s,\ z = t - r + s.$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = \left(\frac{1}{3}r, \frac{1}{3}r - s, t - r + s\right).$$

(*e*) (*i*) Let W be the null space of T.

So W is the set of all (x, y, z) such that

$$T(x, y, z) = (0, 0, 0)$$

i.e., $$(x - 2y - z, y - z, x) = (0, 0, 0)$$

$\Rightarrow$ W is the solution space of

$$x - 2y - z = 0,\ y - z = 0,\ x = 0,$$

which has (0, 0, 0) as a trivial solution.

Thus = {0}.

Hence T is non-singular and hence is invertible.

(*ii*) Let $T(x, y, z) = (r, s, t)$

$\therefore$ $T^{-1}(r, s, t) = (x, y, z)$

Now $T(x, y, z) = (x - 2y - z, y - z, x) = (r, s, t)$

$\Rightarrow$ $x - 2y - z = r,\ y - z = s,\ x = t$

$\Rightarrow$ $$x = t,\ y = \frac{1}{3}(t - r + s),\ z = \frac{1}{3}(t - r - 2s).$$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = \left(t, \frac{1}{3}(t - r + s), \frac{1}{3}(t - r - 2s)\right).$$

Example 18. *If T is a L.T. on V such that $T^2 - T + I = O$, then show that T is invertible.*

(G.N.D.U. 1987)

Sol. We have

$$T^2 - T + I = O$$

$\Rightarrow$ $T^2 = T - I$

$\Rightarrow$ $T^2(\alpha) = (T - I)\alpha$

$\Rightarrow$ $T(T(\alpha)) = T(\alpha) - I(\alpha)$

Let $T(\alpha) = \beta$

$\therefore$ $T(\beta) = \beta - \alpha$

$\Rightarrow$ $\gamma = \beta - \alpha$, where $T(\beta) = \gamma$.

To prove. *T is invertible.*

For this, we prove (*i*) T is one-one, (*ii*) T is onto.

(*i*) *T is one-one.*

Let $T(\alpha_1) = T(\alpha_2)$ $\therefore$ $\beta_1 = \beta_2$ ...(1)

$\Rightarrow$ $T(\beta_1) = T(\beta_2)$ $\Rightarrow$ $\gamma_1 = \gamma_2$

$\Rightarrow$ $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$ $\Rightarrow$ $-\alpha_1 = -\alpha_2$ [$\because$ of (*1*)]

$\Rightarrow$ $\alpha_1 = \alpha_2$

So $T(\alpha_1) = T(\alpha_2)$ $\Rightarrow$ $\alpha_1 = \alpha_2$.

Thus T is one-one.

(*ii*) *T is onto.*

$\forall\ \beta \in V,\ \exists \gamma \in V$ s.t.

$T(\beta) = \gamma$

But $\gamma = \beta - \alpha$ [*As above*]

$\therefore\quad \forall \beta \in V,\ \exists \beta - \alpha \in V$

$\Rightarrow\quad \forall \beta \in V,\ \exists\ \alpha \in V$, s.t. $T(\alpha) = \beta$.

Thus T is onto.

Hence T is invertible.

Example 19. *Can you give an example of a linear operator T such that $T \neq O$ but $T^2 = O$? Justify your assertion.* (G.N.D.U. 1989)

Sol. Yes.

Reason. Let T be defined as $T(a, b, c) = (0, 0, a)$.

Clearly $T \neq O$.

But $T^2(a, b, c) = T(T(a, b, c)) = T(0, 0, a)$

$= (0, 0, 0) = O.$

Example 20. *Give an example of a linear transformation T on $V_3(R)$ such that $T \neq O,\ T^2 \neq O$ but $T^3 = O$.*

Sol. Let T be defined as

$T(a, b, c) = (0, a, b)$ ...(1)

Clearly $T \neq O$.

Now $T^2(a, b, c) = T(T(a, b, c)) = T(0, a, b)$ [*By (1)*]

$= (0, 0, a)$ [*By (1)*]

Thus $T^2 \neq O$.

Now $T^3(a, b, c) = T^2(T(a, b, c)) = T^2(0, a, b)$ [*By (1)*]

$= T(T(0, a, b)) = T(0, 0, a)$ [*By (1)*]

$= (0, 0, 0)$ [*By (1)*]

Thus $T^3 = O$.

Example 21. *If $T: R^3 \to R^3$ such that $T(a, b, c) = (0, a, b)$, then show that $T \neq O,\ T^2 \neq O$ but $T^3 = O$.*

Sol. Same as Ex. 20.

Example 22. *Let $T_1, T_2 : U \to V$ and $T_3, T_4 : V \to W$.*

Prove the following : (*i*) $T_3(T_1 + T_2) = T_3T_1 + T_3T_2$

(*ii*) $(T_3 + T_4)\,T_1 = T_3T_1 + T_4T_1$

(*iii*) $\alpha\,(T_3\,T_1) = (\alpha T_3)\,T_1 = T_3(\alpha T_1)$.

Sol. Given : $T_1, T_2 : U \to V$ and $T_3, T_4 : V \to W$.

$T_1 + T_2 : U \to V$ and $T_3 + T_4 : V \to W$.

We know that for a composite transformation, the range of post-factor is equal to domain of pre-factor, therefore, T_3T_1, T_3T_2, $T_3(T_1 + T_2)$, T_4T_1, T_4T_2, $T_4(T_1 + T_2)$ are all defined and are linear transformations from V to W.

(*i*) Here $T_3(T_1 + T_2)$, T_3T_1, T_3T_2 are all linear transformations form $U \to W$ $\quad \forall\, x \in U$

$$T_3(T_1 + T_2)(x) = T_3((T_1 + T_2)(x)) = T_3(T_1(x) + T_2(x))$$
$$= (T_3T_1)(x) + (T_3T_2)(x) = (T_3T_1 + T_3T_2)(x)$$

Hence $\quad T_3(T_1 + T_2) = T_3T_1 + T_3T_2.$

(*ii*) $(T_3 + T_4)T_1$, T_3T_1, T_4T_1 are all linear transformations from $V \to W$ $\quad \forall\, x \in U$

$$(T_3 + T_4)T_1(x) = (T_3 + T_4)(T_1(x)) = (T_3T_1)(x) + (T_4T_1)(x)$$
$$= (T_3T_1 + T_4T_1)(x).$$

Hence $\quad (T_3 + T_4)T_1 = T_3T_1 + T_4T_1$.

(*iii*) $\forall\, x \in U$

$$\alpha(T_3T_1)x = \alpha[(T_3T_1)(x)] = \alpha[T_3(T_1(x))] = (\alpha T_3)T_1(x) = ((\alpha T_3)T_1)x$$

Again $\quad T_3(\alpha T_1)x = T_3((\alpha T_1)x) = T_3(\alpha T_1(x)) = \alpha T_3(T_1(x)) = \alpha(T_3T_1)x$

Hence $\quad \alpha(T_3T_1) = (\alpha T_3)T_1 = T_3(\alpha T_1).$

Example 23. *If T_1 and T_2 are linear transformations on a finite dimensional vector space V and if $T_1T_2 = I$, then T_1, T_2 are both invertible and $T_2 = T_1^{-1}$.*

Sol. Given : $T_1T_2 = I$.

(*i*) **To prove.** T_2 is invertible.

(I) $\quad T_2(x_1) = T_2(x_2) \Rightarrow T_1[T_2(x_1)] = T_1[T_2(x_2)]$

$$\Rightarrow T_1T_2(x_1) = T_1T_2(x_2)$$
$$\Rightarrow I(x_1) = I(x_2)$$
$$\Rightarrow x_1 = x_2.$$

Hence T_2 is one-one.

(II) Since T_2 is a linear transformation on a finite dimensional vector space, which is one-one, *i.e.*, non-singular,

$\therefore \quad T_2$ is one-one

Hence T_2 is invertible.

Similarly it can be proved that T_1 is invertible.

(*ii*) Since T_2 is invertible, $\quad \therefore\ T_2T_2^{-1} = T_2^{-1}T_2 = I$ $\quad$ [*Def.*]

Also T_2^{-1} is invertible.

Now $T_1T_2 = I \quad \Rightarrow (T_1T_2)T_2^{-1} = IT_2^{-1}$

$$\Rightarrow T_1(T_2T_2^{-1}) = T_2^{-1} \Rightarrow T_1I = T_2^{-1}$$
$$\Rightarrow T_1 = T_2^{-1}$$
$$\Rightarrow T_1T_2 = T_2^{-1}T_2 = I \text{ and } T_2T_1 = T_2T_2^{-1} = I$$

Thus $\quad T_1T_2 = T_2T_1 = I.$

Hence $\quad T_2 = T_1^{-1}.$

Example 24. *If V(F) be the vector space of all polynomials in t and D and T be two linear transformations in V defined as*

$$D(p(t)) = \frac{dp}{dt}$$

and $\quad T(p(t)) = tp(t) \quad \forall\, p(t) \in V,$

then show that product of these two transformations is not commutative and

$$(TD)^2 = T^2 D^2 + TD.$$

Sol. We have $\mathrm{D}(p(t)) = \dfrac{dp}{dt}$ ...(1)

and $\quad \mathrm{T}(p(t)) = t(p(t))$ ...(2)

(*i*) $\quad (\mathrm{DT})(p)(t) = \mathrm{D}(\mathrm{T}(p(t))) = \mathrm{D}(tp(t)) = \dfrac{d}{dt}(tp(t)) = p(t) + t\dfrac{dp}{dt}$ ...(3)

and $\quad (\mathrm{TD})(p(t)) = \mathrm{TD}(p(t)) = \mathrm{T}\left(\dfrac{dp}{dt}\right)$ [*Using (1)*]

$$= t\frac{dp}{dt} \quad \text{...(4)} \quad [\textit{Using (2)}]$$

Subtracting (4) from (3),

$$(\mathrm{DT})(p(t)) - \mathrm{TD}(p(t)) = p(t)$$

$\Rightarrow \quad (\mathrm{DT} - \mathrm{TD})(p(t)) = \mathrm{I}(p(t))$, where I is identity.

As above identity is true $\forall\, p(t) \in \mathrm{V}$,

so $\quad \mathrm{DT} - \mathrm{TD} = \mathrm{I} \Rightarrow \mathrm{DT} \neq \mathrm{TD}$.

Hence the product of D and T is not commutative.

(*ii*) $\quad (\mathrm{TD})^2(p(t)) = (\mathrm{TD})((\mathrm{TD})p)(t) = (\mathrm{TD})(\mathrm{T}\{\mathrm{D}(p(t))\})$

$$= (\mathrm{TD})\left(t\frac{dp}{dt}\right) \quad [\textit{Using (4)}]$$

$$= \mathrm{T}\left(\frac{d}{dt}\left(\frac{tdp}{dt}\right)\right) = \mathrm{T}\left(\frac{dp}{dt} + t\frac{d^2p}{dt^2}\right)$$

$$= t\frac{dp}{dt} + t^2\frac{d^2p}{dt^2} \quad \text{...(5)} \quad [\textit{Using (2)}]$$

Also $\quad (\mathrm{T}^2\mathrm{D}^2 + \mathrm{TD})(p(t)) = (\mathrm{T}^2\mathrm{D}^2)(p(t)) + (\mathrm{TD})(p(t))$

$$= \mathrm{T}^2(\mathrm{D}^2(p(t))) + t\frac{dp}{dt} \quad [\textit{Using (4)}]$$

$$= \mathrm{T}^2(\mathrm{D}\{\mathrm{D}(p(t))\}) + t\frac{dp}{dt}$$

$$= \mathrm{T}^2\left(\mathrm{D}\left(\frac{dp}{dt}\right)\right) + t\frac{dp}{dt} \quad [\textit{Using (1)}]$$

$\Rightarrow \quad T_2T_1(x_1) = T_2T_1(x_2)$

$\Rightarrow \quad x_1 = x_2$ [$\because$ *T_2T_1, being invertible, is one-one*]

Again T_1T_2 is onto,

$\therefore \quad \forall\, y \in V\ \exists\, x \in V$ s.t.

$y = T_1T_2(x) = T_1[T_2(x)]$.

Since T_2 is an operator on V,

$\therefore \quad T_2(x) = z$; say $\in V$

$\therefore \quad y = T_1(z)$

$\therefore \quad \forall\, y \in V\ \exists\, z \in V$ s.t. $y = T_1(z)$

$\Rightarrow$ T_1 is onto.

Since T_1 is both one-one and onto, $\therefore$ T_1 is invertible.

Similarly T_2 is invertible. **[Do it]**

Example 26. *Let $A = \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$, $B = \{\beta_1, \beta_2, \ldots\ldots, \beta_n\}$ be two ordered bases for a finite dimensional vector space V(F). Prove that there exists a unique invertible linear transformation T on V such that $T(\alpha_i) = \beta_i$.*

Sol. Here we are to show that T is invertible.

We know that if V is finite dimensional vector space, then T is invertible $\Leftrightarrow$ T is non-singular.

Here it will be sufficient to prove that T is non-singular.

i.e., $\quad T(\alpha) = O \Rightarrow \alpha = O.$

Let $\alpha \in V$ and α is a basis for V such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n$$

Now $\quad T(\alpha) = O$

$\Rightarrow \quad T(a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n) = O$

$\Rightarrow \quad a_1 T(\alpha_1) + a_2 T(\alpha_2) + \ldots\ldots + a_n T(\alpha_n) = O$

$\Rightarrow \quad a_1\beta_1 + a_2\beta_2 + \ldots\ldots + a_n\beta_n = O$ [$\because$ $T\alpha_i = \beta_i$]

$\Rightarrow \quad a_1 = 0, a_2 = 0, \ldots\ldots, a_n = 0$ [$\because$ *β is L.I.*]

$\Rightarrow \quad \alpha = 0.$

Thus $T(\alpha) = O \Rightarrow \alpha = 0.$

$\Rightarrow$ T is non-singular

$\Rightarrow$ T is invertible. [$\because$ *V is finite dimensional*]

Example 27. *Let $T_1 : R^3 \to R^2$, $T_2 : R^3 \to R^2$, $T_3 : R^3 \to R^2$ be defined by*

$T_1(x, y, z) = (x + y + z, x + y)$, $T_2(x, y, z) = (2x + z, x + y)$, $T_3(x, y, z) = (2y, x)$. Show that $T_1, T_2, T_3 \in L(R^3, R^2)$ are L.I.

Sol. Suppose, for scalars $\alpha, \beta, \gamma \in R$

$\alpha T_1 + \beta T_2 + \gamma T_3 = O$ (zero mapping)

For $e_1 = (1, 0, 0) \in R_3$, we have

$$(\alpha T_1 + \beta T_2 + \gamma T_3)\, e_1 = \alpha T_1(1, 0, 0) + \beta T_2(1, 0, 0) + \gamma T_3(1, 0, 0)$$
$$= \alpha(1, 1) + \beta(2, 1) + \gamma(0, 1) = (\alpha + 2\beta, \alpha + \beta + \gamma)$$

Its transpose is said to be **matrix of the linear transformation** T w.r.t. bases B and B′ and is written **symbolically** as

$$[\mathrm{T} : \mathrm{B}, \mathrm{B}'] \text{ or as } [\mathrm{T}].$$

Hence $[\mathrm{T}] = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}$

Method. (*i*) *Write $T(x_j)$ for each of the basis number in B in terms of elements of B′.*

(*ii*) *Write the coefficient matrix.* (*iii*) *Take its transpose.*

Particular Case. *When $T : U \to V$ i.e., when T is a linear operator.*

Here V = U and hence $m = n$ so that T w.r.t basis B will be $m \times n$ matrix.

Rule for this is also same as above.

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Theorem I. *To every matrix $[a_{ij}]$ of mn scalars belonging to F, there corresponds a linear transformation T from V into U, where V(F), U(F) are vector spaces of n, m dimension respectively.*

Proof. Since T is a linear transformation from V into U and let

$$\mathrm{B} = \{x_1, x_2, \ldots\ldots, x_n\}, \mathrm{B}' = \{y_1, y_2, \ldots\ldots, y_m\}$$

be ordered bases of V, U respectively.

Then $[\mathrm{T} : \mathrm{B}, \mathrm{B}'] = [a_{ij}]_{n \times m}$,

where $\mathrm{T}(x_j) = \sum_{i=1}^{m} a_{ij} y_i$, where $j = 1, 2, \ldots\ldots, n$ ...(1)

To prove. T(x) is uniquely expressible as linear combination of the elements of B′ $\forall x \in$ V.

Each $x \in$ V is uniquely expressible as

$$x = \sum_{i=1}^{m} \mathrm{B}_j x_j$$

Then
$$\mathrm{T}(x) = \left(\mathrm{T} \sum_{j=1}^{n} \mathrm{B}_j x_j\right) = \sum_{j=1}^{n} \mathrm{B}_j \, \mathrm{T}(x_j) \qquad [\because T \text{ is linear}]$$

$$= \sum_{j=1}^{n} \mathrm{B}_j \sum_{i=1}^{m} \alpha_{ij} \, y_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \mathrm{B}_j\right) y_i$$

$$= \sum_{i=1}^{m} \eta_i \, y_i, \text{ where } \eta_i = \sum_{j=1}^{n} \alpha_{ij} \beta_j$$

$\Rightarrow$ T(x) is uniquely expressible as linear combination of the elements of B′. [$\because$ *B_j's are unique*]

$\Rightarrow$ T(x) is uniquely defined $\forall x \in$ V.

Now $\forall \sum_i \alpha_{ij}\ y_i \in$ U's there is a L.T. T from V into U s.t.

$$T(x_j) = \sum_{i=1}^{m} \alpha_{ij}\, y_i, \quad \text{where } j = 1, 2, \ldots\ldots, n$$

$\Rightarrow$ $\exists$ L.T. from V into U as defined by (1) corresponding to each matrix $[\alpha_{ij}]$.

Hence $[x, \mathrm{B}]$ is co-ordinate matrix of x relative to B and

$[\mathrm{T}(x), \mathrm{B}']$ is co-ordinate matrix of $\mathrm{T}(x)$ relative to B'.

Theorem II. *Let $B = \{x_1, x_2, \ldots\ldots, x_n\}$ be a basis of vector space $V(F)$ and T be a linear transformation on V. Then for any vector $x \in V$*

$$[T, B]\,[x, B] = [T(x), B]$$

Proof. We have $\mathrm{B} = \{x_1, x_2, \ldots\ldots, x_n\}$ is a basis of V(F).

Let $\quad T(x_j) = a_{1j}x_1 + a_{2j}x_2 + \ldots\ldots + a_{nj}x_n$

$$= \sum_{i=1}^{n} a_{ij}\, x_i, \text{ where } j = 1, 2, \ldots\ldots, n \qquad \ldots(1)$$

$\therefore$ [T, B] is $n \times n$ matrix whose jth column is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{nj} \end{bmatrix}$$

$$\forall x \in \mathrm{V}, x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = \sum_{j=1}^{n} \alpha_j\, x_j \qquad \ldots(2)$$

The column vector of x is $[x, \mathrm{B}] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

Now $\quad T(x) = T\left(\sum_{j=1}^{n} \alpha_j x_j\right) \qquad [\because \text{ of } (2)]$

$$= \sum_{j=1}^{n} \alpha_j\, T(x_j) = \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n} a_{ij} x_i\right) \qquad [\because \text{ of } (1)]$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}\alpha_j\right) x_i$$

$$= \sum_{i=1}^{n} (a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots\ldots + a_{in}\alpha_n)\, x_i$$

$$\therefore \text{ By def., } [T(x), B] = \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \dots\dots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \dots\dots + a_{2n}\alpha_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots\dots + a_{nn}\alpha_n \end{bmatrix}$$

$$\text{Now} \quad [T, B][x, B] = \begin{bmatrix} a_{11} & a_{12}\dots\dots a_{1n} \\ a_{21} & a_{22}\dots\dots a_{2n} \\ \dots & \dots\dots\dots \\ a_{n1} & a_{n2} \ \dots a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \dots\dots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \dots\dots + a_{2n}\alpha_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \dots\dots + a_{nn}\alpha_n \end{bmatrix}$$

$$= [T(x), B].$$

Hence the result.

Theorem III. *If T_1 and T_2 be two linear operators on vector space $V(F)$ whose matrices w.r.t. a fixed basis*

$$B = \{x_1, x_2, \dots\dots, x_n\} \text{ be } [T_1] \text{ and } [T_2],$$

then the matrices of the operators $T_1 + T_2$, kT_1 and T_1T_2 w.r.t. basis B can be put in the form :

(*i*) $[T_1 + T_2] = [T_1] + [T_2]$ (*ii*) $[kT_1] = k[T_1]$

(*iii*) $[T_1T_2] = [T_1][T_2]$.

Proof. The basis of V is $B = \{x_1, x_2, \dots\dots, x_n\}$.

Let $[T_1] = [a_{ij}]$ and $T_2 = [b_{ij}]$.

Since $[T_1]$ and $[T_2]$ are linear operators,

$\therefore$ $[T_1]$ and $[T_2]$ are both $n \times n$ matrices

$$\therefore\ T_1(x_j) = \sum_{i=1}^{n} a_{ij}x_i \text{ and } T_2(x_j) = \sum_{i=1}^{n} b_{ij}\, x_i.$$

(*i*) $(T_1 + T_2)(x_j) = T_1(x_j) + T_2(x_j)$ [$\because$ *of linearity*]

$$= \sum_{i=1}^{n} a_{ij}x_i + \sum_{i=1}^{n} b_{ij}\, x_i = \sum_{i=1}^{n} (a_{ij} + b_{ij})\, x_i$$

$$= \sum c_{ij}\, x_i, \text{ where } c_{ij} = a_{ij} + b_{ij}$$

$$\therefore \quad [T_1 + T_2] = [c_{ij}] = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}] = [T_1] + [T_2].$$

(*ii*) $(kT_1)(x_j) = kT_1(x_j)$, where $j = 1, 2, \dots\dots, n$; $k \in F$

$$= k\left(\sum_{i=1}^{n} a_{ij}\, x_i\right) = \sum_{i=1}^{n} (ka_{ij})\, x_i$$

Proof. V(F) is n-dimensional vector space, where

$$B = \{x_1, x_2, \ldots\ldots, x_n\} \text{ is an ordered basis of V.}$$

Let T be any L.T. on V, and further let

$$T(x_j) = \sum_{i=1}^{n} a_{ij} x_i\ ,\quad j = 1, 2, \ldots\ldots, n$$

$\therefore$ matrix of T relative to basis B is $[T] = [a_{ij}]$

Let us define a mapping ψ as below :

$$\psi = L(V, V) \to M$$

$$\psi(T) = [T] \in M \quad i.e., \psi(T) = [a_{ij}]$$

Let $\quad [T_1] = [b_{ij}] \quad$ and $\quad [T_2] = [c_{ij}]$,

where $\quad T_1(x_j) = \sum_{i=1}^{n} b_{ij} x_i$ and $T_2(x_j) = \sum_{i=1}^{n} c_{ij} x_i$

(*i*) **To prove.** *ψ is one-one.*

$\forall\ T_1, T_2 \in L(V, V)$,

$$\begin{aligned} \psi(T_1) = \psi(T_2) &\Rightarrow [T_1] = [T_2] \\ &\Rightarrow [b_{ij}] = [c_{ij}] \\ &\Rightarrow b_{ij} = c_{ij}\ ;\ i, j = 1, 2, 3, \ldots\ldots, n \\ &\Rightarrow \sum_{i=1}^{n} b_{ij} x_i = \sum_{i=1}^{n} c_{ij} x_i \\ &\Rightarrow T_1(x_j) = T_2(x_j)\ \forall\ x_j \in B \\ &\Rightarrow T_1 = T_2. \end{aligned}$$

Thus ψ is one-one.

(*ii*) **To prove.** *ψ is onto.*

$\forall\ [a_{ij}] \in M$, there exists a L.T. $T \in L(V, V)$ s.t.

$$T(x_j) = \sum_{i=1}^{n} a_{ij} x_i$$

$\Rightarrow \qquad [T] = [a_{ij}]$

$\Rightarrow \qquad \psi[T] = [T] = [a_{ij}]$

$\therefore \qquad \psi$ is onto.

(*iii*) **To prove.** *ψ is linear.*

$\forall\ T_1, T_2 \in L(V, V)$ and $\alpha, \beta \in F$

$\alpha T_1 + \beta T_2 \in L(V, V)$ $\qquad$ [$\because$ *L(V, V) is a vector space*]

$$\therefore \quad \psi(\alpha T_1 + \beta T_2) = [\alpha T_1 + \beta T_2] = [\alpha T_1] + [\beta T_2] = \alpha[T_1] + \beta[T_2]$$

$$= \alpha\,\psi(T_1) + \beta\,\psi(T_2)$$

Thus ψ is linear.

Combining (*i*), (*ii*) and (*iii*), we have

$$L(V, V) \simeq M.$$

Theorem VI. Matrix of an inverse operator.

Let $V[F]$ be n-dimensional vector space $B = \{x_1, x_2, \ldots\ldots, x_n\}$ as its basis. Let a linear operator T be defined whose matrix relative to B is $[T] = [a_{ij}]$. Then

T is non-singular $\Leftrightarrow$ *[T] is non-singular and hence* $[T^{-1}] = [T]^{-1} = [a_{ij}]^{-1}$.

Proof. Let T be non-singular *i.e.*, T is invertible.

$\therefore$ $\exists$ an inverse operator T^{-1} on V such that

$$TT^{-1} = I = T^{-1}T$$

$\Rightarrow$ $[TT^{-1}] = [I] = [T^{-1}T]$

$\Rightarrow$ $[T][T^{-1}] = [I] = [T^{-1}][T]$

$\Rightarrow$ [T] is non-singular and hence $[T]^{-1} = [T^{-1}]$.

Hence $[T^{-1}] = [T]^{-1} = [a_{ij}]^{-1}$, which is true.

Conversely. Since $[T]^{-1}$ is a matrix.

$\therefore$ $\exists$ a linear transformation T_1 on V such that

$$[T_1] = [T]^{-1}$$

$\Rightarrow$ $[T_1][T] = [I] = [T][T_1]$

$\Rightarrow$ $[T_1T] = [I] = [TT_1]$

$\Rightarrow$ $T_1T = I = TT_1$

$\Rightarrow$ T is invertible

$\Rightarrow$ T is non-singular.

Theorem VII. *Let V(F) and W(F) be two finite dimensional vector spaces of dimension n and m respectively and T_1 and T_2 be any two transformations from V into W. If $v \in V$, then*

(*i*) $[T_1(v)] = [T_1][v]$ (*ii*) $[T_1 + T_2] = [T_1] + [T_2]$

(*iii*) $[\alpha T_1] = \alpha [T_1], \alpha \in F$.

Proof. Let $B_1 = \{v_1, v_2, \ldots\ldots, v_n\}$ be the ordered basis of V

and $B_2 = \{w_1, w_2, \ldots\ldots, w_m\}$ be the ordered basis of W.

Let $T_1(v_{ij}) = \sum_{i=1}^{m} a_{ij}\, w_i$, where $j = 1, 2, \ldots, n$

$\therefore$ by def., $[T_1] = [a_{ij}]_{m \times n}$.

Also $T_2(v_{ij}) = \sum_{i=1}^{m} b_{ij}\, w_i$, where $j = 1, 2, \ldots, n$

$\therefore$ by def., $[T_2] = [b_{ij}]_{m \times n}$.

(*i*) Let v be any vector $\in$ V with B_1 as its base.

$\therefore$ v is a linear combination of elements of B

$\Rightarrow$ $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_m$, where α's $\in$ F.

$\therefore$ The vector v is represented by the column matrix as below :

$$[v] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Also $\quad T_1(v) = T_1(\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n)$

$$= T\left(\sum_{j=1}^{n} \alpha_j v_j\right) \qquad [\because T \text{ is linear}]$$

$$= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} a_{ij} w_i\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\alpha_{ij}\right) w_i, \quad \text{where } j = 1, 2, \ldots\ldots, n$$

$$\therefore \quad [T_1(v)] = \left[\sum_{j=1}^{n} a_{ij}\alpha_j\right]_{m \times n}, \text{ where } i = 1, 2, \ldots, m$$

$$= [a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots\ldots + a_{in}\alpha_n]_{m \times 1}, \text{ where } 1 \leq i \leq m$$

$$= \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots\ldots + a_{2n}\alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \ldots\ldots + a_{mn}\alpha_n \end{bmatrix} \qquad \ldots(1)$$

and

$$[T_1][v] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots\ldots + a_{2n}\alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \ldots\ldots + a_{mn}\alpha_n \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), $[T_1(v)] = [T_1][v]$.

(*ii*) Since $T_1 : V \to W$ and $T_2 : V \to W$

$\therefore \quad T_1 + T_2 : V \to W$

$$\therefore \quad (T_1 + T_2)(v_j) = T_1(v_j) + T_2(v_j) = \sum_{i=1}^{m} a_{ij} w_i + \sum_{i=1}^{m} b_{ij} w_i = \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_i$$

$\therefore \quad [T_1 + T_2] = [a_{ij} + b_{ij}]$, where $1 \leq i \leq m$ and $1 \leq j \leq n$

$$= [a_{ij}] + [b_{ij}] = [T_1] + [T_2].$$

(*iii*) Since $T_1 : V \to W$ and $\alpha \in F$

and $$T_1\ (v_j) = \sum_{i=1}^{m} a_{ij} w_i \text{, where } j = 1, 2, \ldots\ldots, n$$

Now $$(\alpha T_1)(v_j) = \alpha\,(T_1 v_j) = \alpha \sum_{i=1}^{m} a_{ij} w_i, \quad \text{where } j = 1, 2, \ldots\ldots, n$$

$$= \sum_{i=1}^{m} (\alpha a_{ij}) w_i$$

$\Rightarrow$ $$[\alpha T_1] = [\alpha a_{ij}], \text{ where } 1 \le i \le m \text{ and } 1 \le j \le n$$

$$= \alpha[a_{ij}] = \alpha[T_1].$$

Theorem VIII. *Let U(F), V (F) and W(F) be three finite dimensional vector spaces of dimensions m, n and p respectively*

$$T_1 : U \to V,\ T_2 : V \to W$$

be two linear transformations, then $[T_2 T_1] = [T_2]\,[T_1]$.

Proof. Let $B_1 = \{u_1, u_2, \ldots\ldots, u_m\}$, $B_2 = \{v_1, v_2, \ldots\ldots, v_n\}$ and

$$B_3 = \{w_1, w_2, \ldots\ldots, w_p\}$$

be the ordered bases of U, V and W respectively.

Since $T_1 : U \to V$,

so let $T_1\ (u_j) = \sum_{i=1}^{n} a_{ij} v_i$, where $j = 1, 2, \ldots\ldots, m$ ...(1)

$\Rightarrow$ $[T_1] = [a_{ij}]_{n \times m}$ *[By def.]*

Again since $T_2 : V \to W$,

so let $T_2\ (v_k) = \sum_{l=1}^{n} b_{lk} w_l$, where $k = 1, 2, \ldots\ldots, n$ ...(2)

$\Rightarrow$ $[T_2] = [b_{lk}]_{p \times n}$

Since $T_1 : U \to V$ and $T_2 : V \to W$,

$\therefore$ $T_2 T_1$ is defined [$\because$ *Range* (T_1) = *Domain* (T_2)]

Also $T_2\,T_1 : U \to W$

$\therefore$ $$(T_2 T_1)\, u_j = T_2\,(T_1 u_j)$$

$$= T_2 \left(\sum_{i=1}^{n} a_{ij} v_i \right), \text{ where } j = 1, 2, \ldots\ldots, m$$

$$= \sum_{i=1}^{n} a_{ij} T_2\,(v_i) \qquad [\because T_2 \text{ is L.T.}]$$

Proof. (*i*) Let $\quad y_i = \sum_{i=1}^{n} a_{ij}\, x_i = a_{1j}x_1 + a_{2j}x_2 + \ldots\ldots + a_{nj}x_n,$

where $\quad j = 1, 2, \ldots\ldots, n.$

Clearly P is $n \times n$ matrix whose jth column is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

Since $B_2 = \{y_1, y_2, \ldots\ldots, y_n\}$ is an ordered basis of V and $v \in V$.

$$\therefore \qquad v = \alpha_1 y_1 + \alpha_2 y_2 + \ldots\ldots + \alpha_n y_n = \sum_{j=1}^{n} \alpha_j y_j$$

$\therefore$ v can be written as

$$[v : B_2] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Thus $$v = \sum_{j=1}^{n} \alpha_j y_j = \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n} a_{ij} x_i \right) \qquad \text{[Putting the value of } y_j]$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \alpha_j \right) x_i = \sum_{i=1}^{n} (a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots\ldots + a_{in}\alpha_n) x_i$$

Clearly $[v : B_1]$ is a column vector whose ith entry is

$$[a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots\ldots + a_{in}\alpha_n]$$

$$\therefore \qquad P\,[v : B_2] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots\ldots + a_{2n}\alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots\ldots + a_{nn}\alpha_n \end{bmatrix}$$

Hence $P\,[v : B_2] = [v : B_1]$.

(*ii*) We have proved in part (*i*) that

$$P\,[v : B_2] = [v : B_1]$$

Pre-multiplying by P^{-1}, we get

$$P^{-1}P\,[v : B_2] = P^{-1}\,[v : B_1] \quad \Rightarrow \quad I\,[v : B_2] = P^{-1}\,[v : B_1]$$

Hence $\qquad [v : B_2] = P^{-1}\,[v : B_1].$

Theorem II. *Let T be the transition matrix from the basis $B_1 = \{x_i\}$ to the basis $B_2 = [y_i]$ in a vector space V. Then for any linear operator T on V,*

$$[T : B_2] = P^{-1}\,[T : B_1]\,P.$$

Proof. We have proved in Th. I that

$$P\,[v : B_2] = [v : B_1] \quad \forall \quad v \in V \qquad \text{...(1)}$$

Pre-multiplying by $P^{-1}\,[T : B_2]$, we get

$$P^{-1}\,[T : B_1]\,P\,[v : B_2] = P^{-1}\,[T : B_1]\,[v : B_1]$$

$$= P^{-1}\,[T(v) : B_1] \qquad \text{...(2)}$$

$$[\because [T : B_1]\,[v : B_1] = [T(v) : B_1]]$$

$$= [T(v) : B_2] \qquad [\textit{Using (1)}]$$

But $\quad [T : B_2]\ [v : B_2] = [T(v) : B_2]$

$\therefore \quad P^{-1}\,[T : B_1]\,P\,[v : B_2] = [T : B_2]\,[v : B_2]$

Hence $\qquad [T : B_2] = P^{-1}\,[T : B_1]\,P.$

15. Similarity

Def. (*i*) **Similar Matrices.** *Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be similar if there exists a non-singular matrix $C = [c_{ij}]$ such that*

$$AC = CB \quad \textit{Or} \quad A = CBC^{-1}.$$

(*ii*) **Similar Transformations.** *Two transformations T_1 and T_2 of a vector space $V(F)$ are said to be similar if there exists a non-singular transformation P on V such that*

$$T_2 = PT_1P^{-1}.$$

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Theorem I. *The relation of similarity is an equivalence relation in the set M_n of all $n \times n$ matrices over a field F.* (Pbi. U. 1986)

Proof. (*i*) Let A be any $n \times n$ matrix over F.

Then there exists an $n \times n$ invertible matrix I s.t.

$$A = IAI^{-1}$$

$\Rightarrow$ A is similar to itself.

Thus the relation of similarity in M_n is reflexive.

(*ii*) Let A, B $\in M_n$ s.t. A is similar B

$\Rightarrow$ there exists an $n \times n$ invertible matrix C s.t.

$A = CBC^{-1}$

$\Rightarrow$ $AC = CB$

$\Rightarrow$ $C^{-1}AC = B$

$\Rightarrow$ $(C^{-1})\,A\,(C^{-1})^{-1} = B$

$\Rightarrow$ B is similar to A $\qquad [\because C^{-1} \textit{ is invertible}]$

Thus the relation of similarity in M_n is symmetric.

(*iii*) Let A, B, C $\in M_n$ s.t. A is similar to B and B is similar to C

$\Rightarrow$ there exist $n \times n$ invertible matrices P and Q s.t.

$A = PBP^{-1}$ and $B = QCQ^{-1}$

$\Rightarrow$ $A = P(QCQ^{-1})P^{-1} = (PQ)C(Q^{-1} . P^{-1}) = (PQ)C(PQ)^{-1}$

$\Rightarrow$ A is similar to C [$\because$ *PQ is invertible*]

Thus the relation of similarity in M_n is transitive.

Combining (*i*), (*ii*) and (*iii*), the relation of similarity in M_n is an equivalence relation.

Theorem II. *Let $B_1 = \{x_1, x_2, \ldots\ldots, x_n\}$ and $B_2 = \{y_1, y_2, \ldots\ldots, y_n\}$ be two bases of vector space $V(F)$. Also let T_1 and T_2 be two linear transformations on $V(F)$ whose matrices relative to B_1 and B_2 are equal i.e.,*

$$[T_1 : B_1] = [T_2 : B_2] = [a_{ij}],$$

then T_2 is similar to T_1.
i.e., there exists a non-singular L.T.P. such that $T = PT_1 P^{-1}$.

Proof. Since $[T_1 : B_1] = [a_{ij}]$,

$\therefore$ $T_i(x_j) = \sum_{i=1}^{n} a_{ij} x_i$, where $j = 1, 2, \ldots\ldots, n$ ...(1)

Since $[T_2 : B_2] = [a_{ij}]$,

$\therefore$ $T_2(y_j) = \sum_{i=1}^{n} a_{ij} y_i$, where $j = 1, 2, \ldots\ldots, n$ ...(2)

Let $[c_{ij}]$ be the transition matrix.

Then $\exists$ a non-singular L.T.P. such that

$$[P : B_1] = [c_{ij}] \qquad \ldots(3)$$

and $$y_j = P(x_j) \qquad \ldots(4)$$

$\therefore$ $$T_2(y_j) = T_2(P(x_j)) = (T_2P)x_j \qquad \ldots(5)$$

Also $$T_2(y_j) = \sum_{i=1}^{n} a_{ij} P(x_j) \qquad [\because \text{ of (2) and (4)}]$$

$$= P\left(\sum_{i=1}^{n} a_{ij} x_j\right) = P(T_1(x_j)) \qquad [\because \text{ of (1)}]$$

$$= (PT_1)x_j$$

From (5) and (6),

$$(T_2P)x_j = (PT_1)x_j, \text{ where } j = 1, 2, \ldots\ldots, n$$

$\Rightarrow$ $T_2P = PT_1$

Post-multiplying by P^{-1},

$$T_2PP^{-1} = PT_1P^{-1}$$

$\Rightarrow$ $T_2I = PT_1P^{-1}$

$\Rightarrow$ $T_2 = PT_1P^{-1}$.

Hence T_2 is similar to T_1 [*By def.*]

SOLVED EXAMPLES

Example 1. *Let $R^2 \to R^2$ be defined by $T(x, y) = (2x - 3y, x + y)$. Compute the matrix of T relative to the basis*
(*i*) $B = \{(1, 0), (0, 1)\}$ (*ii*) $B = \{(1, 2), (2, 3)\}$.

Sol. Let us compute $T(x_j)$, where x_j is the basis element.

(*i*) We have $T(x, y) = (2x - 3y, x + y)$ ...(1)

$$T(x_1) = T(1, 0) = (2.1 - 3.0, 1 + 0) = (2, 1) = 2(1, 0) + 1(0, 1) = 2x_1 + 1x_2 \quad ...(2)$$

$$T(x_2) = T(0, 1) = (2.0 - 3.1, 0 + 1) = (-3, 1) = -3(1, 0) + 1(0, 1)$$
$$= -3x_1 + 1x_2 \quad ...(3)$$

$\therefore$ Coeff. matrix is $\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$.

Thus the matrix of T w.r.t. B is the transpose of the coeff. matrix.

$\therefore$ From (2) and (3).

$$[T : B] = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}.$$

(*ii*) We have $T(x, y) = (2x - 3y, x + y)$ ...(1)

$$T(x_1) = T(1, 2) = (2.1 - 3.2, 1 + 2) = (-4, 3).$$

Let $(-4, 3) = \alpha(1, 2) + \beta(2, 3)$

$\therefore$ $-4 = \alpha + 2\beta$ and $3 = 2\alpha + 3\beta$

Solving $\alpha = 18$ and $\beta = -11$.

Thus $T(x_1) = T(1, 2) = (-4, 3) = 18(1, 2) - 11(2, 3)$ ...(2)

And $T(x_2) = T(2, 3) = (2.2 - 3.3, 2 + 3) = (-5, 5)$

Let $(-5, 5) = \alpha'(1, 2) + \beta'(2, 3)$

$\therefore$ $-5 = \alpha' + 2\beta'$, $5 = 2\alpha' + 3\beta'$

Solving, $\alpha' = 25$ and $\beta' = -15$.

Thus $T(x_2) = T(2, 3) = (-5, 5)$
$= 25(1, 2) + (-15)(2, 3)$ ...(3)

From (2) and (3),

$$[T : B] = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix}.$$

Example 2. *Find the matrix representation of each of the following operators T on R^2 relative to the basis*
(*a*) $B_1 = \{(1, 3), (2, 5)\}$ (*b*) $B_2 = \{(1, 0), (0, 1)\}$
(*i*) $T(x, y) = (2y, 3x - y)$ (*ii*) $T(x, y) = (3x - 4y, x + 5y)$.

Sol. Let us compute $T(x_j)$, where x_j is the basis element.

(*a*) (*i*) We have

$$T(x, y) = (2y, 3x - y) \quad ...(1)$$

$$T(x_1) = T(1, 3) = (2.3, 3.1 - 3) = (6, 0)$$

Let $(6, 0) = \alpha(1, 3) + \beta(2, 5)$

$\therefore$ $6 = \alpha + 2\beta$ and $0 = 3\alpha + 5\beta$

Solving, $\alpha = -30, \beta = 18$.

Thus $\quad T(x_1) = T(1, 3) = (6, 0) = -30(1, 3) + 18(2, 5)$ ...(2)

And $\quad T(x_2) = (2, 5) = (2.5, 3.2 - 5) = (10, 1)$

Let $\quad (10, 1) = \alpha'(1, 3) + \beta'(2, 5)$

$\therefore \quad 10 = \alpha' + 2\beta'$ and $1 = 3\alpha' + 5\beta'$

Solving, $\quad \alpha' = -48, \ \beta' = 29$

Thus $\quad T(x_2) = T(2, 5) = (10, 1) = -48(1, 3) + 29(2, 5)$ ... (3)

From (2) and (3),

$$[T : B_1] = \begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}.$$

(*ii*) We have

$T(x, y) = (3x - 4y, x + 5y)$ ...(1)

$T(x_j) = T(1, 3) = (3.1 - 4.3, 1 + 5.3) = (-9, 16)$

Let $\quad (-9, 16) = \alpha(1, 3) + \beta(2, 5)$

$\therefore \quad -9 = \alpha + 2\beta$ and $16 = 3\alpha + 5\beta$

Solving, $\quad \alpha = 77, \ \beta = -43.$

Thus $\quad T(x_1) = T(1, 3) = (-9, 16)$

$= 77(1, 3) - 43(2, 5)$ ...(2)

And $\quad T(x_2) = T(2, 5) = (3.2 - 4.5, 2 + 5.5) = (-14, 27)$

Let $\quad (-14, 27) = \alpha'(1, 3) + \beta'(2, 5)$

$\therefore \quad -14 = \alpha' + 2\beta'$ and $27 = 3\alpha' + 5\beta'$

Solving, $\quad \alpha' = 124, \ \beta' = -69$

Thus $\quad T(x_2) = T(2, 5) = (-14, 27)$ ...(3)

$= 124(1, 3) - 69(2, 5)$

From (2) and (3),

$$[T : B_2] = \begin{bmatrix} 77 & 124 \\ -43 & -69 \end{bmatrix}.$$

(*b*) (*i*) We have

$T(x, y) = (2y, 3x - y)$ ...(1)

$T(x_1) = T(1, 0) = (2.0, 3.1 - 0) = (0, 3)$

Let $\quad (0, 3) = \alpha(1, 0) + \beta(0, 1)$

$\therefore \quad 0 = \alpha$ and $3 = \beta$

$\Rightarrow \quad \alpha = 0$ and $\beta = 3$

Thus $\quad T(x_1) = T(1, 0) = (0, 3) = 0.(1, 0) + 3.(0, 1)$ ...(2)

And $\quad T(x_2) = T(0, 1) = (2.1, 3.0 - 1) = (2, -1)$

Let $\quad (2, -1) = \alpha'(1, 0) + \beta'(0, 1)$

$\therefore \quad 2 = \alpha'$ and $-1 = \beta'$

$\Rightarrow \quad \alpha' = 2$ and $\beta' = -1$

Thus $T(x_2) = T(0, 1) = (2, -1) = 2(1, 0) - 1(0, 1)$ ...(3)

From (2) and (3),

$$[T : B_1] = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}.$$

(*ii*) We have

$$T(x, y) = (3x - 4y, x + 5y) \quad ...(1)$$

$$T(x_1) = T(1, 0) = (3.1 - 4.0, 1 + 5.0) = (3, 1)$$

Let $(3, 1) = \alpha(1, 0) + \beta(0, 1)$

$\therefore$ $3 = \alpha$ and $1 = \beta$

$\Rightarrow$ $\alpha = 3$ and $\beta = 1$.

Thus $T(x_1) = T(1, 0) = (3, 1) = 3\,(1, 0) + 1(0, 1)$ ...(2)

And $T(x_2) = T(0, 1) = (3.0 - 4.1, 0 + 5.1) = (-4, 5)$

Let $(-4, 5) = \alpha'\,(1, 0) + \beta'\,(0, 1)$

$\therefore$ $-4 = \alpha'$ and $5 = \beta'$

$\Rightarrow$ $\alpha' = -4$ and $\beta' = 5$

Thus $T(x_2) = T(0, 1) = (-4, 5)$

$= -4\,(1, 0) + 5(0, 1)$ ...(3)

From (2) and (3),

$$[T : B] = \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}.$$

Example 3. *Find the matrix representation of each of the following linear mappings relative to the usual basis of R^n.*

(*i*) *$T : R^2 \to R^3$ defined by*

$$T(x, y) = (3x - y, 2x + 4y, 5x - 6y)$$

(*ii*) *$T : R^3 \to R^4$ defined by*

$$T(x, y, z) = (2x + 3y - 8z, x + y + z, 4x - 5z, 6y).$$ (G.N.D.U. 1985 S)

Sol. Let us compute $T(x_j)$, where x_j is the basis element.

(*i*) We have

$$T(x, y) = (3x - y, 2x + 4y, 5x - 6y) \quad ...(1)$$

$$T(x_1) = T(1, 0) = (3.1 - 0, 2.1 + 4.0, 5.1 - 6.0)$$

$$= (3, 2, 5)$$

Let $(3, 2, 5) = \alpha(1, 0, 0) + \beta(0, 1, 0) + \gamma(0, 0, 1)$

$\therefore$ $3 = \alpha,\ 2 = \beta,\ 5 = \gamma.$

Thus $T(x_1) = T(1, 0) = 3(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1)$ ...(2)

And $T(x_2) = T(0, 1) = (3.0 - 1, 2.0 + 4.1, 5.0 - 6.1) = (-1, 4, -6)$

Let $(-1, 4, -6) = \alpha'\,(1, 0, 0) + \beta'\,(0, 1, 0) + \gamma'\,(0, 0, 1)$

$\therefore$ $-1 = \alpha',\ 4 = \beta',\ -6 = \gamma'$

Thus $\quad T(x_2) = -1(1, 0, 0) + 4(0, 1, 0) - 6(0, 0, 1)$ ...(3)

From (2) and (3),

$$[T : B] = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}.$$

(*ii*) We have

$$T(x, y, z) = (2x + 3y - 8z, x + y + z, 4x - 5z, 6y) \quad \text{...(1)}$$

$$T(x_1) = T(1, 0, 0) = (2.1 + 3.0 - 8.0, 1 + 0 + 0, 4.1 - 5.0, 6.0)$$
$$= (2, 1, 4, 0).$$

Let $\quad (2, 1, 4, 0) = \alpha\,(1, 0, 0, 0) + \beta\,(0, 1, 0, 0) + \gamma\,(0, 0, 1, 0) + \delta\,(0, 0, 0, 1)$

$\therefore \quad 2 = \alpha,\ 1 = \beta,\ 4 = \gamma,\ 0 = \delta$

Thus $\quad T(x_1) = T(1, 0, 0, 0) = (2, 1, 4, 0)$

$$= 2\,(1, 0, 0, 0) + 1\,(0, 1, 0, 0) + 4\,(0, 0, 1, 0) + 0.\,(0, 0, 0, 1) \quad \text{...(2)}$$

And $\quad T(x_2) = T(0, 1, 0)$

$$= (2.0 + 3.1 - 8.0, 0 + 1 + 0, 4.0 - 5.0, 6.1)$$
$$= (3, 1, 0, 6)$$

Let $\quad (3, 1, 0, 6) = \alpha'\,(1, 0, 0, 0) + \beta'\,(0, 1, 0, 0) + \gamma'\,(0, 0, 1, 0) + \delta'\,(0, 0, 0, 1)$

$\therefore \quad 3 = \alpha',\ \ 1 = \beta',\ \ 0 = \gamma',\ \ 6 = \delta'$

Thus $\quad T(x_2) = T(0, 1, 0) = (3, 1, 0, 6)$

$$= 3\,(1, 0, 0, 0) + 1\,(0, 1, 0, 0) + 0\,(0, 0, 1, 0) + 6\,(0, 0, 0, 1) \quad \text{...(3)}$$

Lastly $\quad T(x_3) = T(0, 0, 1)$

$$= (2.0 + 3.0 - 8.1, 0 + 0 + 1, 4.0 - 5.1, 6.0)$$
$$= (-8, 1, -5, 0)$$

Let $\quad (-8, 1, -5, 0) = \alpha''\,(1, 0, 0, 0) + \beta''\,(0, 1, 0, 0) + \gamma''\,(0, 0, 1, 0) + \delta''\,(0, 0, 0, 1)$

$\therefore \quad -8 = \alpha'',\ \ 1 = \beta'',\ \ -5 = \gamma'',\ \ 0 = \delta''$

Thus $\quad T(x_3) = T(0, 0, 1) = (-8, 1, -5, 0)$

$$= -8\,(1, 0, 0, 0) + 1\,(0, 1, 0, 0) - 5\,(0, 0, 1, 0) + 0\,(0, 0, 0, 1) \quad \text{...(4)}$$

From (2), (3) and (4),

$$[T : B] = \begin{bmatrix} 2 & 3 & -8 \\ 1 & 1 & 1 \\ 4 & 0 & -5 \\ 0 & 6 & 0 \end{bmatrix}.$$

Example 4. *Let T be the linear operator on R^2 defined by*

$$T(x, y) = (4x - 2y,\ 2x + y).$$

Compute the matrix of T w.r.t. the basis

$$B = \{(1, 1), (-1, 0)\}.$$

Also verify that

$$[T : B]\,[v : B] = [T(v) : B].$$

(P.U. 1993 S, 92)

(*i*) We have $T(x, y, z) = (2y + z, x - 4y, 3x)$

Using (1), $T(1, 1, 1) = (3, -3, 3) = 3\,(1, 1, 1) + (-6)\,(1, 1, 0) + 6\,(1, 0, 0)$

$T(1, 1, 0) = (2, -3, 3) = 3\,(1, 1, 1) + (-6)\,(1, 1, 0) + 5\,(1, 0, 0)$

and $T(1, 0, 0) = 3\,(1, 1, 1) + (-2)\,(1, 1, 0) + (-1)\,(1, 0, 0)$

Hence $$[T : B] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}.$$

(*ii*) Let $(a, b, c) \in R^3$.

Then $v = (a, b, c) = c\,(1, 1, 1) + (b - c)\,(1, 1, 0) + (a - b)\,(1, 0, 0)$

Thus $$[v : B] = \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

And
$$T(v) = T(a, b, c) = (2b + c, a - 4b, 3a)$$
$$= 3a\,(1, 1, 1) + (a - 4b - 3a)\,(1, 1, 0) + (2b + c - a + 4b)\,(1, 0, 0) \qquad [\because \text{ of } (1)]$$
$$= 3a\,(1, 1, 1) + (-2a - 4b)\,(1, 1, 0) + (-a + 6b + c)\,(1, 0, 0)$$

$$\therefore \quad [T(v) : B] = \begin{bmatrix} 3a \\ -2a - 4b \\ -a + 6b + c \end{bmatrix} \qquad \ldots(2)$$

$$\therefore \quad [T : B]\,[v : B] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

$$= \begin{bmatrix} 3a \\ -2a - 4b \\ -a + 6b + c \end{bmatrix} = [T(v) : B] \qquad [\text{By } (2)]$$

Hence $[T : B]\,[v : B] = [T(v) : B]$.

Example 6. *Let T be a linear operator defined by*

$T(x, y, z) = (2y + z, x - 4y, 3x)$.

Find the matrix of T w.r.t. the basis

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Sol. Firstly, we shall find the co-ordinates of an arbitrary vector $(a, b, c) \in R^3$ w.r.t. basis B.

Let $(a, b, c) = \alpha\,(1, 0, 0) + \beta\,(0, 1, 0) + \gamma\,(0, 0, 1) = (\alpha, 0, 0) + (0, \beta, 0) + (0, 0, \gamma)$

$= (\alpha, \beta, \gamma)$

$\Rightarrow \quad \alpha = a, \ \beta = b, \ \gamma = c.$

$\therefore \quad (a, b, c) = a\,(1, 0, 0) + b\,(0, 1, 0) + c\,(0, 0, 1) \qquad \ldots(1)$

We have $T(x, y, z) = (2y + z, x - 4y, 3x)$

Using (1), $T(1, 0, 0) = (0, 1, 3) = 0\,(1, 0, 0) + 1\,(0, 1, 0) + 3\,(0, 0, 1)$

$T(0, 1, 0) = (2, -4, 0) = 2\,(1, 0, 0) + (-4)\,(0, 1, 0) + 0\,(0, 0, 1)$

and $T(0, 0, 1) = (1, 0, 0) = 1\,(1, 0, 0) + 0\,(0, 1, 0) + 0\,(0, 0, 1)$

Hence $$[T : B] = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Example 7. (*a*) *Let* $T : R^3 \to R^2$ *be linear transformation defined by*

$$T(x, y, z) = (2x + y - z, 3x - 2y + 4z).$$

(*i*) *Obtain the matrix of T in the following bases of* R^3 *and* R^2

$$B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

and $$B_2 = \{(1, 3), (1, 4)\}.$$

(*ii*) *Verify that for any vector* $v \in R^3$,

$$[T : B_1, B_2]\,[v, B_1] = [T(v), B_2].$$ (Pbi. U. 1986)

(*b*) (*i*) *Find the matrix representation of the L.T.*

$T : R^2 \to R^3$ *defined by*

$$T(x, y) = (x + 4y, 2x + 3y, 3x - 5y)$$

w.r.t. the ordered bases,

$$B = \{(1, 1), (2, 3)\} \text{ for } R^2$$

and $$B' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\} \text{ for } R^3.$$ (P.U. 1987)

(*ii*) *Find the matrix representation of the L.T.*

$T : R^2 - R^3$ *defined by*

$$T(x, y) = (3x - 2y, 0, x + 4y)$$

w.r.t. ordered bases

$$B = \{(1, 1), (0, 2)\} \text{ for } R^2$$

and $$B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \text{ for } R^3.$$ (G.N.D.U. 1989 ; P.U. 1986)

Sol. (*a*) (*i*) $(a, b) \in \mathrm{R}^2$ and let

$$(a, b) = \alpha(1, 3) + \beta(1, 4) = (\alpha + \beta, 3\alpha + 4\beta)$$

$$\Rightarrow \qquad a = \alpha + \beta,\ b = 3\alpha + 4\beta$$

Solving, $\alpha = 4a - b,\ \beta = b - 3a$

$$\therefore \qquad (a, b) = (4a - b)(1, 3) + (b - 3a)(1, 4) \qquad \ldots(1)$$

Now, $\mathrm{T}(x, y, z) = (2x + y - z, 3x - 2y + 4z)$

$$\therefore \quad \mathrm{T}(1, 1, 1) = (2.1 + 1 - 1, 3.1 - 2.1 + 4.1) = (2, 5)$$

$$= 3\,(1, 3) + (-1)(1, 4) \qquad \ldots(2)$$

[*Here* $a = 2,\ b = 5$]

$$\mathrm{T}(1, 1, 0) = (2.1 + 1 - 0, 3.1 - 2.1 + 4.0) = (3, 1)$$

$$= 11\,(1, 3) + (-8)(1, 4) \qquad \ldots(3)$$

[*Here* $a = 3,\ b = 1$]

and $$\mathrm{T}(1, 0, 0) = (2.1 + 0 - 0, 3.1 - 2.0 + 4.0) = (2, 3)$$

$$= 5\,(1, 3) + (-3)(1, 4) \qquad \ldots(4)$$

[*Here* $a = 2,\ b = 3$]

$$\therefore \ [\mathrm{T} : \mathrm{B}_1 : \mathrm{B}_2] = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$$

(*ii*) If $\quad v=(x,y,z)\in \mathbb{R}^3$

Then proceeding as above, we have

$$v=(x,y,z)=\alpha(1,1,1)+\beta(1,1,0)+\gamma(1,0,0) \quad \ldots(5)$$

$\Rightarrow \quad x=\alpha+\beta+\gamma,\ y=\alpha+\beta,\ z=\alpha$

Solving, $\quad \alpha=z,\ \beta=y-z,\ \gamma=x-y$

Putting in (5), we get

$$v=(x,y,z)=z\,(1,1,1)+(y-z)\,(1,1,0)+(x-y)\,(1,0,0)$$

Also $\quad T(v)=(2x+y-z,\ 3x-2y+4z)$

$$=(8x+4y-4z-3x+2y-4z)\,(1,3)+(3x-2y+4z-6x-3y+3z)\,(1,4)$$

$$=(5x+6y-8z)\,(1,3)+(-3x-5y+7z)\,(1,4)$$

$\therefore \quad [T(v):B_2]=\begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$

Also $\quad [v:B_1]=\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$

$\therefore \quad [T:B_1,B_2]\,[v:B_1]$

$$=\begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}=\begin{bmatrix} 3z+11y-11z+5x-5y \\ -z-8y+8z-3x+3y \end{bmatrix}=\begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$$

$$=[T(v):B_2].$$

Hence $[T:B_1,B_2]\,[v:B_2]=[T(v):B_2]$.

(*b*) (*i*) $(a,b,c)\in\mathbb{R}^3$ and let

$$(a,b,c)=\alpha\,(1,1,1)+\beta\,(1,1,0)+\gamma\,(1,0,0)$$

$\Rightarrow \quad a=\alpha+\beta+\gamma,\ b=\alpha+\beta,\ c=\alpha$

Solving, $\quad \alpha=c,\ \beta=b-c,\ \gamma=a-b$

$\therefore \quad (a,b,c)=c\,(1,1,1)+(b-c)\,(1,1,0)+(a-b)\,(1,0,0) \quad \ldots(1)$

Now, $\quad T(x,y)=(x+4y,\ 2x+3y,\ 3x-5y)$

$\therefore \quad T(1,1)=(1+4.1,\ 2.1+3.1,\ 3.1-5.1)=(5,5,-2)$

$\quad =(-2)\,(1,1,1)+7\,(1,1,0)+0.(1,0,0) \quad$ [*Here* $a=5,\ b=5,\ c=-2$]

and $\quad T(2,3)=(2+4.3,\ 2.2+3.3,\ 3.2-5.3)=(14,13,-9)$

$\quad =(-9)\,(1,1,1)+22\,(1,1,0)+1.\,(1,0,0) \quad$ [*Here* $a=14,\ b=13,\ c=-9$]

$\therefore \quad [T:B,B']=\begin{bmatrix} -2 & -9 \\ 7 & 22 \\ 0 & 1 \end{bmatrix}.$

(*ii*) $(a,b,c)\in\mathbb{R}^3$ and let

$$(a,b,c)=\alpha\,(1,1,0)+\beta\,(1,0,1)+\gamma\,(0,1,1)$$

$\Rightarrow \quad a=\alpha+\beta,\ b=\alpha+\gamma,\ c=\beta+\gamma$

$$= \begin{bmatrix} -7z - 33(y-z) - 13(x-y) \\ 4z + 19(y-z) + 8(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} -13x - 20y + 26z \\ 8x + 11y - 15z \end{bmatrix} = [T(v) : B_2].$$

Hence the verification.

Example 10. *Consider the linear map $\phi : R^2 \to R^3$ defined by*

$$\phi(x_1, x_2) = (3x_1 - 2x_2, 0, x_1 + 4x_2).$$

Find the matrix of ϕ w.r.t. ordered bases

$$B = \{(1, 1), (0, 2)\} \text{ for } R^2$$

and $B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ *for* R^3. (G.N.D.U. 1997, 89 ; P.U. 1986)

Sol. $(a, b, c) \in R^3$ and let

$$(a, b, c) = \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1)$$

$$\Rightarrow \quad a = \alpha + \beta, \ b = \alpha + \gamma, \ c = \beta + \gamma$$

Solving, $\alpha = \dfrac{a+b-c}{2}, \beta = \dfrac{a-b+c}{2}, \gamma = \dfrac{-a+b+c}{2}$

$$\therefore \quad (a, b, c) = \frac{a+b-c}{2}(1,1,0) + \frac{a-b+c}{2}(1,0,1) + \frac{-a+b+c}{2}(0, 1, 1) \quad \ldots(1)$$

Now, $T(x_1, x_2) = (3x_1 - 2x_2, 0, x_1 + 4x_2)$ [*Here* $a = 1, \ b = 0, \ c = 5$]

$$\therefore \quad T(1, 1) = (3 - 2, 0, 1 + 4) = (1, 0, 5)$$

$$= \frac{1+0-5}{2}(1, 1, 0) + \frac{1-0+5}{2}(1, 0, 1) + \frac{-1+0+5}{2}(0, 1, 1)$$

$$= (-2)(1, 1, 0) + 3(1, 0, 1) + 2(0, 1, 1)$$

and $T(0, 2) = (0 - 4, 0, 0 + 8) = (-4, 0, 8)$ [*Here* $a = -4, \ b = 0, \ c = 8$]

$$= \frac{-4+0-8}{2}(1, 1, 0) + \frac{-4-0+8}{2}(1, 0, 1) + \frac{4+0+8}{2}(0, 1, 1)$$

$$= (-6)(1, 1, 0) + 2(1, 0, 1) + 6(0, 1, 1).$$

$$\therefore \quad [T : B, B'] = \begin{bmatrix} -2 & -6 \\ 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

Example 11. *Given the matrix* $\begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$, *determine the corresponding linear operator T on R^2 w.r.t. the basis*

$$B = \{(1, 0), (1, 1)\}.$$ (G.N.D.U. 1988 S)

Sol. We have $[T : B] = \begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$

$\therefore$ Coefficient matrix $= \begin{bmatrix} 1/2 & 2/3 \\ 1 & 4 \end{bmatrix}$

$\therefore \quad T(1, 0) = \frac{1}{2}(1, 0) + \frac{2}{3}(1, 1) = \left(\frac{7}{6}, \frac{2}{3}\right)$

$T(1, 1) = 1\,(1, 0) + 4\,(1, 1) = (5, 4)$

Let $\quad (a, b) = \alpha\,(1, 0) + \beta\,(1, 1) = (\alpha + \beta, \beta)$

$\therefore \quad \alpha + \beta = a,\ \beta = b \qquad \therefore \quad \alpha = a - b.$

$$\therefore \quad T(a, b) = T[(a - b)(1, 0) + b\,(1, 1)]$$

$$= (a - b)\,T(1, 0) + b\,T(1, 1) \qquad [\because T \text{ is linear}]$$

$$= (a - b)\left(\frac{7}{6}, \frac{2}{3}\right) + b\,(5, 4)$$

$$= \left(\frac{7}{6}(a - b) + 5b,\ \frac{2}{3}(a - b) + 4b\right)$$

Hence $\quad T(a, b) = \left(\dfrac{7a + 23b}{6}, \dfrac{2a + 10b}{3}\right).$

Example 12. *Let $V(F)$ be a vector space of all polynomials in x of degree atmost n on a real field and a differentiation transformation D is defined on V as : $D : P_n \to P_n$*

s.t. $\quad D[p(x)] = \dfrac{d}{dx}[p(x)]\ \forall\ p(x) \in V(F).$

Find the matrix of operator D w.r.t. a basis of $V(F)$.

Sol. The basis set is $B = \{1, x, x^2, \ldots\ldots, x^n\}$.

Since D is a linear operator on V,

$\therefore$ We shall find $D(x^p)$, $p = 0, 1, 2, \ldots\ldots, n$ *i.e.*, for each basis number.

$$D(1) = 0 \quad = 0.1 + 0.x + 0.x^2 + \ldots\ldots + 0.x^n$$
$$D(x) = 1 \quad = 1.1 + 0.x + 0.x^2 + \ldots\ldots + 0.x^n$$
$$D(x^2) = 2x \quad = 0.1 + 2x + 0.x^2 + \ldots\ldots + 0.x^n$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$D(x^n) = nx^{n-1} = 0.1 + 0.x + 0.x^2 + \ldots\ldots + nx^{n-1} + 0.x^n$$

$$\therefore \quad \text{Coefficient matrix is } \begin{pmatrix} 0 & 0 & 0 & \ldots\ldots 0 \\ 1 & 0 & 0 & \ldots\ldots 0 \\ 0 & 2 & 0 & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ 0 & 0 & 0 & \ldots n\ 0 \end{pmatrix}$$

$\therefore \quad$ Matrix of operator D = Transpose of coefficient matrix

$$= \begin{pmatrix} 0 & 1 & 0 & \ldots\ldots 0 \\ 0 & 0 & 2 & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots n \\ 0 & 0 & 0 & \ldots\ldots 0 \end{pmatrix}.$$

Example 13. *If $V(F)$ is a vector space of polynomials in t of degree atmost 3 and D be the differentiation transformation on V. Then basis for $V(F)$ is $B = \{1, t, t^2, t^3\}$. Verify that*

$$[D : B]\,[x : B] = [D(x) : B].$$

Sol. Let $x = p(t) = a + bt + ct^2 + dt^3 \in \mathrm{V}$.

Then $$[\mathrm{D} : \mathrm{B}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $$[x : \mathrm{B}] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Since $\mathrm{D}\,(p(t)) = b + 2ct + 3dt^2 + 0.t^3$

$$\therefore \quad [\mathrm{D}\,(p(t)) : \mathrm{B}] = [\mathrm{D}(x) : \mathrm{B}] = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}$$

And $$[\mathrm{D} : \mathrm{B}]\,[x : \mathrm{B}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix} = [\mathrm{D}(x) : \mathrm{B}]$$

Hence $[\mathrm{D} : \mathrm{B}]\,[x : \mathrm{B}] = [\mathrm{D}(x) : \mathrm{B}]$.

Example 14. *If the matrix of the linear transformation T on R^2 relative to usual basis of R is*

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}.$$

Then find the matrix of T relative to the basis

$$B_1 = \{(1, 1), (1, -1)\}.$$

Sol. The usual basis of R^2 is $\mathrm{B} = \{(1, 0), (0, 1)\}$.

Also we have $[\mathrm{T} : \mathrm{B}] = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

$$\therefore \quad \mathrm{T}(1, 0) = 2\,(1, 0) + 1\,(0, 1) = (2, 1)$$

$$\mathrm{T}(0, 1) = -3\,(1, 0) + 1\,(0, 1) = (-3, 1)$$

$$\therefore \quad \mathrm{T}(a, b) = \mathrm{T}(a\,(1, 0) + b\,(0, 1)) = a\mathrm{T}(1, 0) + b\mathrm{T}(0, 1)$$

$$= a\,(2, 1) + b\,(-3, 1) = (2a - 3b, a + b).$$

This defines T on R^2.

The basis of R^2 is $\mathrm{B}_1 = \{(1, 1), (1, -1)\}$

and $$\mathrm{T}(a, b) = (2a - 3b, a + b).$$

We have $\mathrm{T}(1, 1) = (-1, 2) = \dfrac{1}{2}(1, 1) - \dfrac{3}{2}\,(1, -1)$

and $\quad T(1,-1)=(5,0)=\frac{5}{2}(1,1)+\frac{5}{2}(1,-1).$

Hence, by def., $[T:B_1]=\begin{bmatrix} 1/2 & 5/2 \\ -3/2 & 5/2 \end{bmatrix}.$

Example 15. *If the matrix of linear operator T on R^2 relative to standard basis is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, what is the matrix of T relative to the basis $B_1=\{(1,1),(1,-1)\}$.*

Sol. The usual standard basis of R^2 is $B=\{(1,0),(0,1)\}$.

Also we have

$$[T:B]=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\therefore \quad T(1,0)=1(1,0)+1(0,1)=(1,1)$

$\quad T(0,1)=1(1,0)+1(0,1)=(1,1)$

$\therefore \quad T(a,b)=T(a(1,0)+b(0,1))=aT(1,0)+bT(0,1)$

$\quad =a(1,1)+b(1,1)=(a+b,a+b)$

This defines T on R^2.

The basis of R^2 is $B_1=\{(1,1),(1,-1)\}$

and $\quad T(a,b)=(a+b,a+b).$

We have $T(1,1)=(2,2)=\alpha(1,1)+\beta(1,-1)$

$\Rightarrow \quad 2=\alpha+\beta$ and $2=\alpha-\beta$

$\Rightarrow \quad 2\alpha=4 \quad \Rightarrow \alpha=2$ and $\beta=0$

$\therefore \quad T(1,1)=(2,2)=2(1,1)+0(1,-1)$

and $\quad T(1,-1)=(0,0)=\alpha(1,1)+\beta(1,-1)$

$\Rightarrow \quad 0=\alpha+\beta$ and $0=\alpha-\beta$

$\Rightarrow \quad \alpha=\beta=0$

$\therefore \quad T(1,-1)=(0,0)=0(1,1)+1(1,-1).$

Hence, by def., $[T:B_1]=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$

Example 16. *Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let T be the linear operator on R^2 defined by $T(v)=Av$, where v is written as a column vector.*

Find the matrix of T in each of the following bases :

(*i*) $B_1=\{(1,0),(0,1)\}$ (*ii*) $B_2=\{(1,3),(2,5)\}$.

Sol. (*i*) $\quad T(1,0)=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}=1(1,0)+3(0,1)$

$\quad T(0,1)=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \end{bmatrix}=2(1,0)+4(0,1)$

Thus $[T:B]=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(*ii*) Let $v=(a, b) \in R^2$

and $v=(a, b)=\alpha(1, 3)+\beta(2, 5)=(\alpha+2\beta, 3\alpha+5\beta)$

$\therefore$ $a=\alpha+2\beta$ and $b=3\alpha+5\beta$

Solving, $\alpha=2b-5a, \ \beta=3a-b$

$\therefore$ $(a, b)=(2b-5a)(1, 3)+(3a-b)(2, 5)$ ...(1)

$$T(1, 3)=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}=\begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

$=-5(1, 3)+6(2, 5)$ [*Putting* $a=7, \ b=15$ *in* (*1*)]

$$T(2, 5)=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}=\begin{bmatrix} 12 \\ 26 \end{bmatrix}$$

$=-8(1, 3)+10(2, 5)$ [*Putting* $a=12, b=26$ *in* (*1*)]

Hence $[T:B_1]=\begin{bmatrix} -5 & -8 \\ 6 & 10 \end{bmatrix}$.

Example 17. (*a*) *If the matrix of a linear transformation T on* R^3 *relative to the basis* $B=\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *is*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

then what is the matrix of T relative to the basis

$B_1=\{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$. (G.N.D.U. 1988 ; P.U. 1985)

(*b*) *If the matrix of a linear transformation T on* R^3 *relative to the usual basis is*

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

then what is the matrix of T relative to the basis

$B=\{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$. (P.U. 1997, 89 ; Pbi. U. 1997)

Sol. (*a*) The basis is given as

$B=\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

First we define T when

$$[T:B]=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \quad ...(1)$$

Now $T((1, 0, 0))=0(1, 0, 0)+1(0, 1, 0)+(-1)(0, 0, 1)$

$=(0, 1, -1)$ [$\because \alpha=0, \ \beta=1, \ \gamma=-1$ *by* (*1*)]

Example 19. *Each of the sets*

(*i*) $B_1 = \{1, t, e^t, te^t\}$ (*ii*) $B_2 = \{1, t, \sin 3t, \cos 3t\}$

is a basis of a vector space V of function $f : R \to R$. *Let D be the differential operator of V i.e.,* $D(f) = \dfrac{df}{dt}$.

Find the matrix of D in the given basis.

Sol. (*i*)
$$D(1) = 0 = 0.1 + 0.t + 0.e^t + 0.te^t$$
$$D(t) = 1 = 1.1 + 0.t + 0.e^t + 0.te^t$$
$$D(e^t) = e^t = 0.1 + 0.t + 1.e^t + 0.te^t$$
$$D(te^t) = 1.e^t + te^t = 0.1 + 0.t + 1.e^t + 1.te^t$$

Hence $[D : B_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(*ii*)
$$D(1) = 0 = 0.1 + 0.t + 0.\sin 3t + 0.\cos 3t$$
$$D(t) = 1 = 1.1 + 0.t + 0.\sin 3t + 0.\cos 3t$$
$$D(\sin 3t) = 3\cos 3t = 0.1 + 0.t + 0.\sin 3t + 3.\cos 3t$$
$$D(\cos 3t) = -3\sin 3t = 0.1 + 0.t - 3.\sin 3t + 0.\cos 3t$$

Hence $[D : B_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

Example 20. *Consider the vector space V(F) of all 2 × 2 matrices and let T be a linear transformation on V(F) such that*

$$T(x) = Ax, \text{ where } x \in V(F) \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find the matrix of T relative to basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of V (F).

Sol. Given. $T : V \to V$; $T(x) = Ax \ \forall x \in V$ ...(1)

Now $T(x_1) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ [*By* (*1*)]

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$T(x_2) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ [*By* (*1*)]

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T(E_3) = ME_3 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q & 0 \\ s & 0 \end{bmatrix}$$

$$= q\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$T(E_4) = ME_4 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $$[T\,;B] = \begin{bmatrix} p & 0 & r & 0 \\ 0 & p & 0 & r \\ q & 0 & s & 0 \\ 0 & q & 0 & s \end{bmatrix}' = \begin{bmatrix} p & 0 & q & 0 \\ 0 & p & 0 & q \\ r & 0 & s & 0 \\ 0 & r & 0 & s \end{bmatrix}.$$

(*b*) We have $T\,; V \to V$ is a linear operator defined by
$T(A) = AM \;\; \forall\, A \in V.$

where $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$

To find the matrix of T relative to basis

$$B = \{E_1, E_2, E_3, E_4\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

Here $$T(E_1) = E_1M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$$

$$= p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T(E_2) = E_2M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T(E_3) = E_3M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p & q \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + p\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$T(E_4) = E_4M = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r & s \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $$[T\,;B] = \begin{bmatrix} p & q & 0 & 0 \\ r & s & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & r & s \end{bmatrix}' = \begin{bmatrix} p & r & 0 & 0 \\ q & s & 0 & 0 \\ 0 & 0 & p & r \\ 0 & 0 & q & s \end{bmatrix}.$$

(*d*) We have $T : V \to V$ is a linear operator defined by

$T(A) = MA + AM \;\; \forall A \in V$,

where $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

To find the matrix of T relative to basis

$$B = \{E_1, E_2, E_3, E_4\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

Here $T(E_1) = ME_1 + E_1M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2p & q \\ r & 0 \end{bmatrix},$$

$$T(E_2) = ME_2 + E_2M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & p \\ 0 & r \end{bmatrix} + \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} r & p+s \\ 0 & r \end{bmatrix},$$

$$T(E_3) = ME_3 + E_3M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} q & 0 \\ s & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p & q \end{bmatrix}$$

$$= \begin{bmatrix} q & 0 \\ s+p & q \end{bmatrix}$$

and $T(E_4) = ME_4 + E_4M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r & s \end{bmatrix}$

$$= \begin{bmatrix} 0 & q \\ r & 2s \end{bmatrix}.$$

Thus $T(E_1) = \begin{bmatrix} 2p & q \\ r & 0 \end{bmatrix} = 2p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$T(E_2) = \begin{bmatrix} r & p+s \\ 0 & r \end{bmatrix},$$

$$T(E_3) = \begin{bmatrix} q & 0 \\ s+p & q \end{bmatrix} = q\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (s+p)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $T(E_4) = \begin{bmatrix} 0 & q \\ r & 2s \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2s\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\therefore \quad [T\,;B] = \begin{bmatrix} 2p & q & r & 0 \\ r & p+s & 0 & r \\ q & 0 & p+s & q \\ 0 & q & r & 2s \end{bmatrix}^t = \begin{bmatrix} 2p & r & q & 0 \\ q & p+s & 0 & q \\ r & 0 & p+s & r \\ 0 & r & q & 2s \end{bmatrix}.$$

$$[T^{-1}(x)]_B = [T^{-1}]_B [x : B] = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & -2 \\ 3 & -6 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4a+2b-c \\ 8a+13b-2c \\ 3a-6b+3c \end{bmatrix}$$

$$\therefore \qquad T^{-1}(x) = T^{-1}(a, b, c) = \frac{1}{9}(4a+2b-c, 8a+13b-2c, -3a-6b+3c).$$

Example 26. *If T_1, T_2 are similar linear transformations on a finite dimensional vector space $V(F)$, then prove that*

$$det\,[T_1] = det\,[T_2].$$

Sol. Since T_1 and T_2 are similar transformations hence $\exists$ an invertible operator T such that

$$T_1 = TT_2T^{-1}$$

$$\begin{aligned} \therefore \qquad \det[T_1] &= \det[TT_2T^{-1}] = (\det T)(\det T_2)(\det T^{-1}) \\ &= (\det T)(\det T^{-1})(\det T_2) = \det(TT^{-1}) \det T_2 \\ &= (\det I) \det T_2 \\ &= 1 \det T_2 = \det T_2. \end{aligned}$$

Example 27. *If two linear transformations on $V(F)$ are similar, then show that T_1^2 and T_2^2 are also similar and if T_1, T_2 are invertible, then T_1^{-1} and T_2^{-1} are also similar.*

Sol. Since T_1, T_2 are similar transformations hence $\exists$ an invertible transformation T such that

$$T_1 = TT_2T^{-1} \qquad \ldots(1)$$

Now $\quad T_1^2 = T_1T_1 = (TT_2T^{-1})(TT_2T^{-1}) = TT_2(T^{-1}T)T_2T^{-1}$

$\quad = TT_2IT_2T^{-1} \qquad [\because T^{-1}T = TT^{-1} = I]$

$\quad = TT_2T_2T^{-1} = TT_2^2T^{-1}$

Now $\quad T_1^2 = TT_2^2T^{-1} \qquad \therefore T_1^2$ similar to T_2^2.

Now if T_1, T_2 be invertible, then from (1),

$$(T_1)^{-1} = (TT_2T^{-1})^{-1} = (T^{-1})^{-1}T_2^{-1}T^{-1} = TT_2^{-1}T^{-1}.$$

Above relation shows that T_1^{-1} and T_2^{-1} are similar.

Example 28. *If T_1 and T_2 are linear transformations on $V(F)$ and if at least one of them is invertible, then T_1T_2 and T_2T_1 are similar.*

Sol. Let T_1 be invertible, then

$$T_1T_1^{-1} = I \qquad \ldots(1)$$

Now $\quad T_1T_2 = T_1T_2I = T_1T_2T_1T_1^{-1}$

$\Rightarrow \quad T_1T_2 = T_1(T_2T_1)T_1^{-1}$

By def., T_1T_2 and T_2T_1 are similar.

Similarly, we can show by taking T_2 as invertible.

16. TRACE AND DETERMINANT

Definition. *The trace of a square matrix $A = [a_{ij}]$ of order n over a field F is the sum of the elements on the principal diagonal of A.*

We shall write the trace of A as tr A.

Thus $\qquad$ **tr A** $= \sum_{i=1}^{n} a_{ij}$.

THEOREMS

Theorem I. *Let A and B be square matrices of order n over a field F and $\alpha \in F$. Then*

(i) $tr(\alpha A) = \alpha \, tr\, A$ (ii) $tr(A+B) = tr\, A + tr\, B$

(iii) $tr(AB) = tr(BA)$. (G.N.D.U. 1992)

Proof. (i) Let $A = [a_{ij}]$, then $\alpha A = [\alpha a_{ij}]$.

Now $$\operatorname{tr} \alpha A = \sum_{i=1}^{n} \alpha a_{ii} = \alpha \sum_{i=1}^{n} a_{ii}$$

Hence $\operatorname{tr}(\alpha A) = \alpha \operatorname{tr} A$.

(ii) Let $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} (a_{ii} + b_{ii})$$

$$\Rightarrow \quad \operatorname{tr}(A+B) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

Hence $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$.

(iii) Let $AB = [\alpha_{ij}]$ and $BA = [\beta_{ij}]$,

where $\alpha_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ and $\beta_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$

Thus $$\operatorname{tr}(AB) = \sum_{i=1}^{n} \alpha_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}\, b_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right) = \sum_{k=1}^{n} \beta_{kk}$$

Hence $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Cor. *If A and C are square matrices with A invertible, then $tr(A^{-1}CA) = tr\, C$.*

Proof. Let $B = A^{-1}C$, then $\operatorname{tr}(A^{-1}CA) = \operatorname{tr}(BA) = \operatorname{tr}(AB) = \operatorname{tr}(AA^{-1}C)$

$\therefore \quad \operatorname{tr}(A^{-1}CA) = \operatorname{tr} C$.

Theorem II. *Let V be a n-dimensional vector space over F. If B and B′ are two ordered bases of V and $T \in A(V)$, then $tr[T; B] = tr[T; B']$.*

Proof. By Art. 13 ; Th. II, there exists a invertible matrix P such that

$$[T; B'] = P^{-1}[T; B]\, P$$

Thus by above Cor., $\operatorname{tr}[T; B'] = \operatorname{tr}[T; B]$.

Remark. *This theorem shows that trace of $[T; B]$ depends on T and not on any particular ordered bases B of V.*

Definition. *Let V be a finite dimensional vector space over F. If $T \in L(V)$, then* **the trace of** *T is defined as trace of a matrix $[T; B]$, where B is some ordered basis of V.*

Theorem III. *Let V be a finite dimensional vector space over F. If $T, S \in L(V)$ and $\alpha \in F$, then*

(i) $tr(T+S) = tr\, T + tr\, S$ (ii) $tr(\alpha T) = \alpha \, tr\, T$ (iii) $tr(TS) = tr(ST)$.

Proof. Let $B' = \{x_1, x_2, \ldots\ldots, x_n\}$ be a basis of V.

If $\quad A = [T\,;B]$ and $B = [S\,;B']$, then

$$[T + S\,;B'] = A + B$$

and $$[\alpha T\,;B'] = \alpha A$$

Thus $$\text{tr } T = \text{tr } A,\ \text{tr } S = \text{tr } B,\ \text{tr }(T + S) = \text{tr }(A + B)$$

and $$\text{tr }(\alpha T) = \text{tr }(\alpha A).$$

(*i*) $$\text{tr }(T + S) = \text{tr }(A + B) = \text{tr } A + \text{tr } B$$

$$\text{tr }(T + S) = \text{tr } T + \text{tr } S.$$

(*ii*) $$\text{tr }(\alpha T) = \text{tr }(\alpha A) = \alpha \text{ tr }(A)$$

$$\text{tr }(\alpha T) = \alpha \text{ tr } T.$$

(*iii*) $$\text{tr }(TS) = \text{tr }[TS, B'] = \text{tr }([T, B']\,[S, B'])$$

$$= \text{tr }([S, B']\,[T, B']) = \text{tr }([ST, B'])$$

$$\text{tr } TS = \text{tr } ST.$$

SOLVED EXAMPLE

Example. *Let $T : R^3 \to R^3$ defined by*

$$T(a, b, c) = (a + c,\ 2a - 2b,\ -a + b - c)$$

be a L.T.

(*i*) *Find the trace and determinant of T.*

(*ii*) *Is T invertible ? If so, find its inverse.*

Sol. Let $B = \{e_1, e_2, e_3\}$ be a standard basis of R^3. We find $[T\,;B]$

$$T(e_1) = (1, 2, -1) = 1e_1 + 2e_2 - e_3$$

$$T(e_2) = (0, -2, 1) = 0e_1 - 2e_2 + e_3$$

$$T(e_3) = (1, 0, -1) = 1e_1 + 0e_2 - e_3$$

$\therefore$ $$[T\,;B] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Now $$\text{tr } T = \text{tr }[T\,;B] = 1 - 2 - 1 = -2$$

or $$\text{tr } T = -2.$$

$$\text{det. } T = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -2 & 0 \\ -1 & 1 & -1 \end{vmatrix} = 1\,(2) - 0 + 1\,(2 - 2) = 2$$

$$\text{det. } T = 2 \neq 0$$

Thus T is invertible.

Since $\quad T(e_1) = (1, 2, -1), \quad T(e_2) = (0, -2, 1), \quad T(e_3) = (1, 0, -1)$

$\therefore$ $$T^{-1}(1, 2, -1) = (1, 0, 0),$$

$$T^{-1}(0, -2, 1) = (0, 1, 0)$$

and $$T^{-1}(1, 0, -1) = (0, 0, 1)$$

To find $T^{-1}(a, b, c)$. We write (a, b, c) as a linear combination of $(1, 2, -1)$, $(0, -2, 1)$ and $(1, 0, -1)$.

Let $x\,(1, 2, -1) + y\,(0, -2, 1) + z\,(1, 0, -1) = (a, b, c)$ $\quad\ldots(1)$

Augmented matrix of this equation is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 2 & -2 & 0 & b \\ -1 & 1 & -1 & c \end{array}\right]$$

$$\underset{\sim}{R_2 - 2R_1,\ R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -2 & -2 & b - 2a \\ 0 & 1 & 0 & a + c \end{array}\right]$$

$$\underset{\sim}{R_{23}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & a + c \\ 0 & -2 & -2 & b - 2a \end{array}\right]$$

$$\underset{\sim}{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & a + c \\ 0 & 1 & 1 & a - \frac{b}{2} \end{array}\right]$$

$$\underset{\sim}{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & a + c \\ 0 & 0 & 1 & -\frac{b}{2} - c \end{array}\right]$$

$$\underset{\sim}{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a + \frac{b}{2} + c \\ 0 & 1 & 0 & a + c \\ 0 & 0 & 1 & -\frac{b}{2} - c \end{array}\right]$$

Thus, $x = a + \dfrac{b}{2} + c$, $y = a + c$, $z = -\dfrac{b}{2} + c$ is a solution of (1).

Operating T^{-1} on (1), we get

$$xT^{-1}(1, 2, -1) + yT^{-1}(0, -2, 1) + zT^{-1}(1, 0, -1) = T^{-1}(a, b, c)$$

$$\Rightarrow \quad T^{-1}(a, b, c) = xe_1 + ye_2 + ze_3$$

$$\Rightarrow \quad T^{-1}(a, b, c) = \left(a + \frac{b}{2} + c,\ a + c,\ \frac{-b}{2} - c\right).$$

To prove. $\phi(f(T)) = f(A)$.

ϕ is a linear transformation.

Here let T_1 and T_2 be two linear operators on V and let A_1 and A_2 be their matrix representations.

Thus $[T_1] = A$ and $[T_2] = A_2$

$\Rightarrow$ $[T_1 + T_2] = [T_1] + [T_2] = A_1 + A_2$

Also $[\alpha T_1] = \alpha [T_1] = \alpha A_1$

$\Rightarrow$ Matrix representation of αT_1 is αA_1.

Now $\phi(T_1 + T_2) = [T_1 + T_2] = A_1 + A_2 = \phi(T_1) + \phi(T_2)$ and $\phi(\alpha T_1) = [\alpha T_1] = \alpha A_1 = \alpha\phi(T_1)$

Thus ϕ is a linear transformation.

$\therefore$ $\phi(T_1 T_2) = [T_1 T_2] = [T_1][T_2] = \phi(T_1)\phi(T_2)$.

Let $f(x) = a_0 + a_1 x + \ldots\ldots + a_n x^n$.

We prove by induction.

Suppose $n = 0$.

We know that matrix of identity operator I′ is the unit matrix

i.e., $\phi(I') = I$

$\Rightarrow$ $\phi(f(T)) = \phi(f(I'))$

But $(f(I')) = a_0 I'$ $\quad [\because \textit{deg } (f(x)) = 0, \ \therefore f(x) = a_0]$

$\Rightarrow$ $\phi(f(T)) = \phi(f(I'))$

$$= \phi(a_0 I') = a_0 \phi(I') = a_0 I = f(A)$$

$\Rightarrow$ $\phi(f(T)) = f(A)$.

Thus the result is true when $n = 0$.

Let us assume that the result is true for polynomials of degree $< n$.

Then $\phi(f(T_1)) = \phi(a_0 I' + a_1 T + \ldots\ldots + a_{n-1} T^{n-1} + a_n T^n)$

$$= \phi(a_0 I' + a_1 T + \ldots\ldots + a_{n-1} T^{n-1}) + a_n\ \phi(T^n)$$

$$= \phi(a_0 I' + a_1 T + \ldots\ldots + a_{n-1} T^{n-1}) + a_n\ \phi(T)\,\phi(T^{n-1})$$

$$= \left(a_0 I + a_1 A + \ldots\ldots + a_{n-1} A^{n-1}\right) + a_n A A^{n-1}$$

$$= a_0 I + a_1 A + \ldots\ldots + a_{n-1} A^{n-1} + a_n A^n$$

$$= f(A).$$

Thus the result is true for any n.

Hence the result is true for all polynomials.

3. Eigen Values and Eigen Vectors

Defs. (*i*) **Eigen Value.** *Let $T : V \to V$ be the linear operator on a vector space V over F. Then a scalar $\lambda \in F$ is said to be an* **eigen value** *if there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$.*

Eigen values are also known as **Characteristic values or latent values** or **proper values** or **spectral values.**

(*ii*) **Eigen vector.** *Let $T : V \to V$ be the linear operator on a vector space V over F. A scalar $\lambda \in F$, being the eigen value, satisfies $T(v) = \lambda v$, where v is a non-zero vector $\in V$. Any vector satisfying this relation is said to be* **eigen vector** *of T belonging to the eigen value λ.*

Eigen vectors are also known as **characteristic vectors** or **latent vectors** or **proper vectors** or **spectral vectors.**

Theorem IV. *Let λ be an eigen value of a linear operator $T : V \to V$. Let V_λ be the set of all eigen vectors of T belonging to the eigen value λ. Show that V_λ is a sub-space of V.* (P.U. 1995)

Proof. Let $v, w \in V_\lambda$.

By def., $T(v) = \lambda v$ and $T(w) = \lambda w$ ...(1)

For any scalars $\alpha, \beta \in F$, we have

$$T(\alpha v + \beta w) = \alpha T(v) + \beta T(w) = \alpha (\lambda v) + \alpha (\lambda v) \qquad [\textit{Using } (1)]$$

$$= \lambda (\alpha v + \beta w)$$

$\Rightarrow$ $\alpha v + \beta w$ is an eigen vector belonging to eigen value λ

$\Rightarrow$ $\alpha v + \beta w \in V_\lambda$.

Hence V_λ is a sub-space of V.

Note. V_λ, being a sub-space of V, is said to be **eigen space** of λ.

Theorem V. *Let $T : V \to V$ be a linear operator on a vector space V over F. Then $\lambda \in F$ is an eigen value of T iff the operator $\lambda I - T$ is singular. Also the eigen space of λ will be the null space of $\lambda I - T$.*

(P.U. 1996, 88, 85 ; G.N.D.U. 1995 S)

Proof. (*i*) $\lambda \in F$ is an eigen value of T

$\Leftrightarrow$ $\exists$ a non-zero vector v s.t. $T(v) = \lambda v$

$\Leftrightarrow$ $T(v) = \lambda I(v)$ $[\because I(v) = v]$

$\Leftrightarrow$ $(\lambda I) - T(v) = O$

$\Leftrightarrow$ $(\lambda I - T)(v) = O$, where v is non-zero vector

$\Leftrightarrow$ $\lambda I - T$ is singular.

Hence the result.

(*ii*) λ is an eigen value of T, if $\exists$ a non-zero vector v s.t.

$$(\lambda I - T)(v) = 0.$$

Clearly v belongs to the null space of $\lambda I - T$.

Also v is an eigen vector of T belonging to the eigen value λ.

Thus v is in the eigen space of λ iff v is in the null space of $\lambda I - T$.

Hence the null space of $\lambda I - T$ is the eigen space of λ.

Cor. The eigen values of T are given by

$$\det(\lambda I - A) = 0, \text{ where } A = [T].$$

Theorem VI. *Non-zero eigen vectors belonging to distinct values are linearly independent.*

(G.N.D.U. 1996 ; P.U. 1989 ; Pbi. U. 1986)

Proof. Let $v_1, v_2, \ldots\ldots, v_n$ be n non-zero eigen vectors of a linear operator $T : V \to V$ belonging to distinct eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ respectively.

To prove. $v_1, v_2, \ldots\ldots, v_n$ are L.I.

We prove the result by induction.

When $n = 1$. Here v_1 is L.I. because $v_1 \neq 0$.

Thus the result is true when $n = 1$.

Let us assume that the result is true when the number of vectors $< n$.

Let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$ ...(1),

where $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

$\Rightarrow \quad \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = T(0)$ [*Applying T*]

$\Rightarrow \quad \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = 0$

Since $T(v_i) = \lambda_i v_i$ for $i = 1, 2, \ldots\ldots, n$

$\therefore \quad \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \ldots\ldots + a_n\lambda_n v_n = 0$ ...(2)

Multiplying (1) by λ_n, we get

$$\alpha_1 \lambda_n v_1 + \alpha_2 \lambda_n v_2 + \ldots\ldots + a_n\lambda_n v_n = 0 \quad \ldots(3)$$

Subtracting (3) from (2), we get

$$\alpha_1 (\lambda_1 - \lambda_n) v_1 + \alpha_2 (\lambda_2 - \lambda_n) v_2 + \ldots\ldots + \alpha_{n-1} (\lambda_{n-1} - \lambda_n) v_{n-1} = 0$$

$\Rightarrow \quad \alpha_1 (\lambda_1 - \lambda_n) = 0,\ \alpha_2 (\lambda_2 - \lambda_n) = 0,\ \ldots\ldots,\ \alpha_{n-1} (\lambda_{n-1} - \lambda_n) = 0$ ...(4)

[$\because$ $v_1, v_2, \ldots\ldots, v_{n-1}$ *are L.I. (assumed)*]

But since λ_i are distinct,

$\therefore \quad \lambda_1 - \lambda_n \neq 0, \lambda_2 - \lambda_n \neq 0, \ldots\ldots, \lambda_{n-1} - \lambda_n \neq 0.$

$\therefore$ (4) $\Rightarrow$ $\alpha_1 = 0,\ \alpha_2 = 0, \ldots\ldots, \alpha_{n-1} = 0.$

Putting in (1), $\quad \alpha_n v_n = 0$

$\Rightarrow \quad \alpha_n = 0.$ [$\because$ $v_n \neq 0$]

Thus $\alpha_1 = 0,\ \alpha_2 = 0, \ldots\ldots, \alpha_n = 0.$

Hence the vectors $v_1, v_2, \ldots\ldots, v_n$ are L.I.

Theorem VII. *0 is an eigen value of an operator T iff T is singular.* (P.U. 1996)

Proof. 0 is an eigen value of T

$\Leftrightarrow$ there exists a non-zero vector $v \in V$

s.t. $T(v) = 0.v$ *i.e.,* $T(v) = 0$

$\Leftrightarrow$ T is singular.

Theorem VIII. *If* λ *is an eigen value of an invertible operator T on a vector space V over F, then* λ^{-1} *is an eigen value of* T^{-1}. (P.U. 1997 ; G.N.D.U. 1997)

Proof. Since T is invertible, [*Given*]

$\therefore$ T is non-singular

$\therefore$ there exists an eigen value $\lambda \neq 0$

$\Rightarrow$ λ^{-1} exists

Sol. (*i*) Here $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$

$$\therefore \quad A^2 = A.A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8 & -2-10 \\ 4+20 & -8+25 \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix}.$$

Since $f(t) = t^2 - 3t + 7$, [*Given*]

$\therefore \quad f(A) = A^2 - 3A + 7I_2$, where I_2 is the 2-rowed unit matrix

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} + \begin{bmatrix} -3 & 6 \\ -12 & -15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7-3+7 & -12+6+0 \\ 24-12+0 & 17-15+7 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix}.$$

(*ii*) Here $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}.$$

Since $f(t) = 2t^2 - 3t + 7$, [*Given*]

$\therefore \quad f(A) = 2A^2 - 3A + 7I_2$, where I_2 is the 2-rowed unit matrix

$$= 2\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 14-3+7 & 20-6+0 \\ 30-9+0 & 44-12+7 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 21 & 39 \end{bmatrix}.$$

(*iii*) Here $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}.$$

Since $f(t) = t^2 - 5t - 2$,

$\therefore \quad f(A) = A^2 - 5A - 2I_2$, where I_2 is the 2-rowed unit matrix

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} -5 & -10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 7-5-2 & 10-10+0 \\ 15-15+0 & 22-20-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Example 2. *Prove that*

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \text{ is a zero of } f(t) = t^2 - 4t - 5.$$

Sol. Here $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}.$$

Since $f(t) = t^2 - 4t - 5$,

$\therefore \quad f(A) = A^2 - 4A - 5I_2$, where I_2 is 2-rowed unit matrix

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} + \begin{bmatrix} -4 & -16 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16+0 \\ 8-8+0 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Hence A is a zero of $f(t)$.

Example 3. *Find the eigen values and associated non-zero eigen vectors of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

(P.U. 1996)

Sol. Let us assume that there is a scalar λ and a non-zero vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$, such that $AX = \lambda X$.

i.e.,
$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.,
$$\begin{bmatrix} x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.$$

This is equivalent to system of linear homogeneous equations

$$x + 2y = \lambda x \text{ and } 3x + 2y = \lambda y$$

i.e.,
$$\left.\begin{aligned} (\lambda - 1)\, x - 2y &= 0 \\ -3x + (\lambda - 2) y &= 0 \end{aligned}\right] \qquad \ldots(1)$$

We know that a system of linear homogeneous equations has a non-zero solution iff determinant of the co-efficient matrix = 0

iff $\quad \det \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{bmatrix} = 0$

iff $\quad (\lambda - 1)(\lambda - 2) - 6 = 0$

iff $\quad \lambda^2 - 3\lambda + 2 - 6 = 0 \quad$ iff $\quad \lambda^2 - 3\lambda - 4 = 0$

iff $\quad (\lambda - 4)(\lambda + 1) = 0 \quad$ iff $\quad \lambda = 4, -1.$

Hence the eigen values of A are $\lambda = 4$ or $\lambda = -1$.

(I) *When* $\lambda = 4$.

From (1), $\left.\begin{aligned}(4-1)\,x - 2y &= 0\\ -3x + (4-2)\,y &= 0\end{aligned}\right\} \Rightarrow 3x - 2y = 0.$

We easily take $x = 2,\ y = 3$.

Thus $v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a non-zero eigen vector belonging to the eigen value $\lambda = 4$ and any other eigen vector belonging to the eigen value $\lambda = 4$ is a scalar multiple of v.

(II) *When* $\lambda = -1$.

From (1), $\left.\begin{aligned}(-1-1)\,x - 2y &= 0\\ -3x + (-1-2)\,y &= 0\end{aligned}\right\} \Rightarrow x + y = 0.$

We easily take $x = 1,\ y = -1$.

Thus $w = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a non-zero eigen vector belonging to the eigen value $\lambda = -1$ and any other eigen vector belonging to the eigen value $\lambda = -1$ is a scalar multiple of w.

Example 4. (*a*) *In the following matrices, obtain all eigen values and associated L.I. eigen vectors.*

(*i*) $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ (*ii*) $B = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$.

(*b*) *Obtain invertible matrices* P_1, P_2, P_3 *respectively such that* $P_1^{-1}AP_1$, $P_2^{-1}BP_2$ *and* $P_3^{-1}CP_3$ *are diagonal matrices.*

Sol. (*i*) (*a*) Let us assume that there is a scalar λ and non-zero vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$, such that $AX = \lambda X$

i.e., $$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \quad i.e., \quad \begin{bmatrix} 4x + 2y \\ 3x + 3y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.$$

This is equivalent to the system of linear homogeneous equations

$$4x + 2y = \lambda x \text{ and } 3x + 3y = \lambda y$$

i.e., $$\left.\begin{aligned}(\lambda - 4)\,x - 2y &= 0\\ -3x + (\lambda - 3)\,y &= 0\end{aligned}\right] \qquad \ldots(1)$$

We know that a system of linear homogeneous equations has a non-zero solution iff determinant of the co-efficient matrix = 0

iff $\det \begin{bmatrix} \lambda - 4 & -2 \\ -3 & \lambda - 3 \end{bmatrix} = 0$

iff $(\lambda - 4)(\lambda - 3) - 6 = 0$ iff $\lambda^2 - 7\lambda + 12 - 6 = 0$

iff $\lambda^2 - 7\lambda + 6 = 0$ iff $(\lambda - 1)(\lambda - 6) = 0$

iff $\lambda = 1, 6$.

Hence the eigen values of A are $\lambda = 1$ or $\lambda = 6$.

(I) *When* $\lambda = 1$.

From (1), $\left.\begin{aligned}(1-4)\,x - 2y &= 0\\ -3x + (1-3)\,y &= 0\end{aligned}\right] \Rightarrow -3x - 2y = 0 \Rightarrow 3x + 2y = 0.$

We easily take $x = 2,\ y = -3$.

This is equivalent to the system of linear homogeneous equations

$$5x - y = \lambda x \quad \text{and} \quad x + 3y = \lambda y$$

i.e., $$\left.\begin{aligned} (\lambda - 5)x + y &= 0 \\ -x + (\lambda - 3)y &= 0 \end{aligned}\right] \qquad \ldots(1)$$

We know that a system of linear homogeneous equations has a non-zero solution iff determinant of the co-efficient matrix = 0

iff $\det \begin{bmatrix} \lambda - 5 & 1 \\ -1 & \lambda - 3 \end{bmatrix} = 0$

iff $(\lambda - 5)(\lambda - 3) + 1 = 0$ iff $\lambda^2 - 8\lambda + 15 + 1 = 0$

iff $\lambda^2 - 8\lambda + 16 = 0$ iff $(\lambda - 4)^2 = 0$

iff $\lambda = 4, 4$

Hence the eigen value of A is $\lambda = 4$.

When $\lambda = 4$.

From (1), $$\left.\begin{aligned} (4 - 5)x + y &= 0 \\ -x + (4 - 3)y &= 0 \end{aligned}\right] \Rightarrow -x + y = 0 \Rightarrow x - y = 0.$$

We easily take $x = 1,\ y = 1$.

Thus $v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a non-zero eigen vector belonging to the eigen value $\lambda = 4$ and any other eigen vector belonging to the eigen value $\lambda = 4$ is a scalar multiple of v.

(*b*) P_2 does not exist because B has only one L.I. eigen vector and as such can't be diagonalized.

Example 5. *Find the eigen values for the matrix*

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

Sol. Let us assume that there is a scalar λ and a non-zero vector

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ such that } AX = \lambda X$$

i.e., $$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} 3x + y + z \\ 2x + 4y + 2z \\ x + y + 3z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}.$$

This is equivalent to system of linear homogeneous equations

$$\begin{aligned} 3x + y + z &= \lambda x \\ 2x + 4y + 2z &= \lambda y \\ x + y + 3z &= \lambda z \end{aligned}$$

i.e., $$\left.\begin{aligned} (3 - \lambda)x + y + z &= 0 \\ 2x + (4 - \lambda)y + 2z &= 0 \\ x + y + (3 - \lambda)z &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

We know that a system of linear homogeneous equations has a non-zero solution

iff $$\det\begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix} = 0$$

iff $$\det\begin{bmatrix} 3-\lambda & 0 & 1 \\ 2 & 2-\lambda & 2 \\ 1 & -2+\lambda & 3-\lambda \end{bmatrix} = 0 \qquad [\textit{Operating } C_2 \to C_2 - C_3]$$

iff $$(\lambda-2)\det\begin{bmatrix} 3-\lambda & 0 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix} = 0$$

iff $$(\lambda-2)\det\begin{bmatrix} 3-\lambda & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 0 & 5-\lambda \end{bmatrix} = 0 \qquad [\textit{Operating } R_3 \to R_3 + R_2]$$

iff $$(\lambda-2)(-1)\det\begin{bmatrix} 3-\lambda & 1 \\ 3 & 5-\lambda \end{bmatrix} = 0 \qquad [\textit{Expanding by } C_2]$$

iff $-(\lambda-2)[(3-\lambda)(5-\lambda)-(3)(1)] = 0$

iff $-(\lambda-2)(15-3\lambda-5\lambda+\lambda^2-3) = 0$

iff $-(\lambda-2)(\lambda^2-8\lambda+12) = 0$

iff $-(\lambda-2)(\lambda-2)(\lambda-6) = 0$

iff $-(\lambda-2)^2(\lambda-6) = 0$ iff $\lambda = 2, 2, 6.$

Hence 2, 6 are eigen values.

Example 6. (*a*) *In the following matrices, obtain all eigen values and a basis for the corresponding eigen space*

(*i*) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$ (G.N.D.U. 1995 S)

(*ii*) $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(*b*) *If possible, obtain invertible matrices P_1 and P_2 respectively such that $P_1^{-1}AP_1$ and $P_2^{-1}BP_2$ are diagonal matrices.*

Sol. (*i*) (*a*) Let us assume that there is a scalar λ and a non-zero vector.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ such that } AX = \lambda X$$

i.e., $$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \textit{i.e.,} \quad \begin{bmatrix} x+2y+2z \\ x+2y-z \\ -x+y+4z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

Thus $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Hence eigen spaces are (1, 1, 0) and (1, 0, 1).

(*b*) A has three L.I. eigen vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{and} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

corresponding to eigen values $\lambda = 3, 3, 1$.

Let $\quad P_1 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

To find P_1^{-1}:

$$\text{adj } P_1 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 3 & -1 \end{bmatrix}^t = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

and $\quad \det P_1 = \det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$= 1\,(0+1) - 1(1-2) = 1 + 1 = 2 \neq 0$$

$$\therefore \quad P_1^{-1} = \frac{\text{adj } P_1}{\det P_1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Then A is similar to the diagonal matrix

$$P_1^{-1}\, AP_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 0 & \frac{1}{2} \\ 0 & 3 & 0 \\ 4 & 0 & -1 \end{bmatrix}. \qquad [\textit{Verify !}]$$

(*ii*) (*a*) Let us assume that there is a scalar λ and a non-zero vector

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ such that } BX = \lambda X$$

i.e., $\quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x+y \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}.$

This is equivalent to system of linear homogeneous equations

$$\left.\begin{aligned} x + y &= \lambda x \\ y &= \lambda y \\ \text{and} \qquad z &= \lambda z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} (1-\lambda)\, x + y &= 0 \\ (1-\lambda)\, y &= 0 \\ (1-\lambda)\, z &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

We know that a system of linear homogeneous equations has a non-zero solution

iff $\quad \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

iff $\quad (1-\lambda)^3 = 0 \qquad$ iff $\quad \lambda = 1, 1, 1.$

Hence 1 is the eigen value.

When $\lambda = 1$, from (1), $\left.\begin{aligned} 0.x + y &= 0 \\ 0.y &= 0 \\ 0.z &= 0 \end{aligned}\right\} \Rightarrow y = 0$

Clearly eigen spaces are (1, 0, 0) and (0, 0, 1).

Example 7. *Find all the eigen values and a basis of each eigen space of the linear operator* $T : R^2 \to R^3$ *defined by*

$$T(x, y) = (3x + 3y, x + 5y). \qquad \text{(P.U. 1993 S ; G.N.D.U. 1985 S)}$$

Sol. First of all, let us find a matrix representation of T ; say relative to the usual basis

$B = \{(1, 0), (0, 1)\}$ of R^2

$A = [T] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}.$

The eigen values of T are the values of λ s.t.

$\det [\lambda I - A] = 0$

i.e., $\quad \det \begin{bmatrix} \lambda - 3 & -3 \\ -1 & \lambda - 5 \end{bmatrix} = 0$

i.e., $\quad (\lambda - 3)(\lambda - 5) - 3 = 0 \quad$ *i.e.,* $\quad \lambda^2 - 8\lambda + 15 - 3 = 0$

i.e., $\quad \lambda^2 - 8\lambda + 12 = 0 \quad$ *i.e.,* $\quad (\lambda - 2)(\lambda - 6) = 0.$

Thus 2 and 6 are eigen values of T.

(*i*) *Basis of the eigen space of eigen value 2.*

Putting $\lambda = 2$ in $(\lambda I - A) X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x-3y \\ -x-3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x - 3y = 0, \qquad -x - 3y = 0$

i.e. $\quad x + 3y = 0, \qquad x + 3y = 0.$

The system has an independent solution $x = 3,\ y = -1$.

Thus $\alpha_1 = (3, -1)$ forms a basis.

(*ii*) *Basis of the eigen space of eigen value 6.*

Putting $\lambda = 6$ in $(\lambda I - A) = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x-3y \\ -x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 3x - 3y = 0, \qquad -x + y = 0$

i.e., $\quad x - y = 0, \qquad x - y = 0.$

$\therefore$ The system has an independent solution $x = 1,\ y = 1.$

Thus $\alpha_2 = (1, 1)$ forms a basis.

Example 8. *In the following operators $T : R^2 \to R^2$, find all the eigen values and basis for the corresponding eigen space*

(*i*) $T(x, y) = (y, x)$ (G.N.D.U. 1985 S)

(*ii*) $T(x, y) = (y, -x)$ (*iii*) $T(x, y) = (3x + 3y, x + 5y)$. (P.U. 1997)

Sol. (*i*) First of all, let us find a matrix representation of T ; say relative to the usual basis

$B = \{(1, 0), (0, 1)\}$ of R^2

$$A = [T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigen values of T are the values of λ s.t. det $[\lambda I - A] = 0$

i.e., $\det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = 0$ *i.e.* $\lambda^2 - 1 = 0$ *i.e.* $\lambda = \pm 1.$

Thus $1, -1$ are eigen values of T.

(*I*) *Basis of eigen space of eigen value 1.*

Putting $\lambda = 1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x - y = 0,\ -x + y = 0.$

The system has an independent solution $x = 1,\ y = 1.$

Hence $\alpha_1 = (0, 0)$ forms a basis.

(*II*) *Basis of eigen space of eigen value – 1.*

Putting $\lambda = -1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x - y \\ -x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x - y = 0, \quad -x - y = 0.$

The system has an independent solution $x = 1,\ y = 1.$

Hence $\alpha_2 = (1, -1)$ forms a basis.

(*ii*) First of all, let us find a matrix representation of T ; say relative to the usual basis

$B = \{(1, 0), (0, 1)\}$ of R^2.

$$A = [T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigen values of T are the values of λ s.t. det $[\lambda I - A] = 0$

i.e., $\det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0,$

which has no solution in R.

Hence there is no eigen value in R.

The eigen values of T are the values of λ s.t. det $[\lambda I - A] = 0$

i.e., $$\det \begin{bmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 2 & -1 \\ 0 & -2 & \lambda - 3 \end{bmatrix} = 0$$

i.e., $(\lambda - 1)[(\lambda - 2)(\lambda - 3) - 2] = 0$ $\Rightarrow$ $(\lambda - 1)(\lambda^2 - 5\lambda + 6 - 2) = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0$ $\Rightarrow$ $(\lambda - 1)(\lambda - 1)(\lambda - 4) = 0$

$\Rightarrow$ $(\lambda - 1)^2(\lambda - 4) = 0$ $\Rightarrow$ $\lambda = 1, 1, 4.$

Thus 1 and 4 are eigen values of T.

(*I*) *Basis of the eigen space of eigen value 1.*

Putting $\lambda = 1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -y-z \\ -y-z \\ -2y-2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $-y - z = 0,\ -y - z = 0,\ -2y - 2z = 0$

or $y = 0,\ z = 0,\ x$ can have any value.

Thus $\alpha_1 = (1, 0, 0)$ forms a basis.

(*II*) *Basis of the eigen space of value 4.*

Putting $\lambda = 4$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x - y - z \\ 2y - z \\ -2y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $3x - y - z = 0, 2y - z = 0, -2y + z = 0$

$\therefore$ $3x - y - z = 0,\ 2y = z.$

The system has an independent solution $x = 1,\ y = 1,\ z = 2.$

Hence $\alpha_2 = (1, 1, 2)$ forms a basis.

Example 10. *In the following operators* $T : R^3 \to R^3$, *find all eigen values and a basis for the corresponding eigen space.*

(*i*) $T(x, y, z) = (2x + y, y - z, 2y + 4z)$ (G.N.D.U. 1985)

(*ii*) $T(x, y, z) = (x + y, y + z, -2y - z)$ (G.N.D.U. 1987)

(*iii*) $T(x, y, z) = (x + y + z, 2y + z, 2y + 3z)$

(*iv*) $T(x, y, z) = (x - y, 2x + 3y + 2z, x + y + 2z).$

Sol. (*i*) First of all, let us find a matrix representation of T ; say relative to the usual basis

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of R^3

$$A = [T] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

The eigen values of T are the values of λ s.t. det $[\lambda I - A] = 0$

i.e., $$\det\begin{bmatrix}\lambda-2 & -1 & 0\\ 0 & \lambda-1 & 1\\ 0 & -2 & \lambda-4\end{bmatrix}=0$$

i.e., $$(\lambda-2)\det\begin{bmatrix}\lambda-1 & 1\\ -2 & \lambda-4\end{bmatrix}=0$$

$\Rightarrow$ $(\lambda-2)[(\lambda-1)(\lambda-4)+2]=0$ $\Rightarrow$ $(\lambda-2)[\lambda^2-5\lambda+6]=0$

$\Rightarrow$ $(\lambda-2)(\lambda-2)(\lambda-3)=0$ $\Rightarrow$ $(\lambda-2)^2(\lambda-3)=0$

$\Rightarrow$ $\lambda = 2, 2, 3.$

Hence 2 and 3 are eigen values of T.

(*I*) *Basis of the eigen space of eigen value 2.*

Putting $\lambda = 2$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & -2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix}-y\\ y+z\\ -2y-2z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\therefore$ $-y=0, y+z=0, -2y-2z=0$

$\Rightarrow$ $y=0, z=0, x$ can have any value.

Thus $\alpha_1 = (1, 0, 0)$ forms a basis.

(*II*) *Basis of the eigen space of eigen value 3.*

Putting $\lambda = 3$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 2 & 1\\ 0 & -2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix}x-y\\ 2y+z\\ -2y-z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\therefore$ $x-y=0, 2y+z=0, -2y-z=0$

$\Rightarrow$ $x-y=0, 2y+z=0$ $\Rightarrow$ $x=y, z=-2y$

$$\therefore\quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}y\\ y\\ -2y\end{bmatrix}=\begin{bmatrix}1\\ 1\\ -2\end{bmatrix}y.$$

Hence $\alpha_2 = (1, 1, -2)$ forms a basis.

(*ii*) First of all, let us find a matrix representation of T ; say relative to the usual basis

$$B = \{(1,0,0), (0,1,0), (0,0,1)\} \text{ of } R^3$$

$$A = [T] = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & -2 & -1\end{bmatrix}$$

The eigen values of T are the values of λ s.t. det. $[\lambda I - A] = 0$

i.e., $$\det\begin{bmatrix}\lambda-1 & -1 & 0\\ 0 & \lambda-1 & -1\\ 0 & 2 & \lambda+1\end{bmatrix}=0$$

$\Rightarrow \quad (\lambda - 1)\det\begin{bmatrix} \lambda-1 & -1 \\ 2 & \lambda+1 \end{bmatrix} = 0 \quad \Rightarrow (\lambda - 1)[(\lambda - 1)(\lambda + 1) + 2] = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 - 1 + 2) = 0 \qquad \Rightarrow \quad (\lambda - 1)(\lambda^2 + 1) = 0$

$\Rightarrow \quad \lambda = 1.$ $\qquad [\because \lambda^2 + 1 \neq 0 \text{ in } R]$

Basis of the eigen space of eigen value 1.

Putting $\lambda = 1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \begin{bmatrix} -y \\ -z \\ 2y+2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow -y = 0, -z = 0, 2y + 2z = 0$

$\therefore \quad y = 0, z = 0$, and x can have any value.

Thus $\alpha = (1, 0, 0)$ can be taken as basis.

(*iii*) First of all, let us find a matrix representative of T ; say relative to the usual basis

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ of } R^3$$

$$A = [T] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

The eigen values of T are the values of λ s.t. $\det[\lambda I - A] = 0$

i.e., $\quad \det\begin{bmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda-2 & -1 \\ 0 & -2 & \lambda-3 \end{bmatrix} = 0$

$\Rightarrow \quad (\lambda - 1)\det\begin{bmatrix} \lambda-2 & -1 \\ -2 & \lambda-3 \end{bmatrix} = 0 \quad \Rightarrow (\lambda - 1)[(\lambda - 2)(\lambda - 3) - 2] = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0 \qquad \Rightarrow \quad (\lambda - 1)(\lambda - 4)(\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 1, 4, 1.$

Thus 1, 4 are two eigen values of T.

(I) ***Basis of the eigen space of eigen value 1.***

Putting $\lambda = 1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_2 \to R_2 - R_1 \textit{ and } R_3 \to R_3 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} -y-z \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow -y-z=0 \qquad \Rightarrow y=-z$$

and x can have any non-zero value.

$\therefore \quad \alpha_1 = (1, 1, -1)$.

(II) ***Basis of eigen space of eigen value 4.***

Putting $\lambda = 4$ in $(\lambda I - X) = O$, we get the homogeneous system of equations.

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_3 \to R_3 + R_2]$$

$$\Rightarrow \begin{bmatrix} 3x-y-z \\ 2y-z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow 3x-y-z=0 \text{ and } 2y-z=0$$

$\Rightarrow \quad x = y, z = 2y$

$\therefore \quad y$ is a free variable $\qquad \therefore \quad \alpha_2 = (1, 1, 2)$.

(*iv*) First of all, let us find a matrix representative of T ; say relative to the usual basis

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ of } R^3$$

$$A = [T] = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

The eigen values of T are the values of λ s.t. $\det[\lambda I - A] = 0$

i.e.,
$$\det \begin{bmatrix} \lambda-1 & 1 & 0 \\ -2 & \lambda-3 & -2 \\ -1 & -1 & \lambda-2 \end{bmatrix} = 0$$

$$\Rightarrow \det \begin{bmatrix} \lambda-1 & 1 & 0 \\ -2 & \lambda-3 & -2 \\ \lambda-2 & 0 & \lambda-2 \end{bmatrix} = 0 \qquad [\textit{Operating } R_3 \to R_3 + R_1]$$

$$\Rightarrow (\lambda-2) \det \begin{bmatrix} \lambda-1 & 1 & 0 \\ -2 & \lambda-3 & -2 \\ 1 & 0 & 1 \end{bmatrix} = 0$$

$\Rightarrow \quad (\lambda-2)\left[(\lambda-1)(\lambda-3)-1(-2+2)\right]=0$ [*Expanding by R_1*]

$\Rightarrow \quad (\lambda-1)(\lambda-2)(\lambda-3)=0 \qquad \Rightarrow \lambda=1,2,3.$

Thus 1, 2, 3 are eigen values of T.

(I) ***Basis of the eigen space of eigen value 1.***

Putting $\lambda = 1$ in $(\lambda I - A)X = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 0 & 1 & 0 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_2 \to R_2 - 2R_3]$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_1 \leftrightarrow R_2]$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_1 \leftrightarrow R_2]$$

$$\Rightarrow \begin{bmatrix} -x-y-z \\ y \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow -x-y-z=0, y=0$$

$\Rightarrow \quad x=-y-z, y=0 \qquad \Rightarrow x=-z, y=0$

$\therefore \quad \alpha_1=(1,0,-1)$

(II) ***Basis of eigen space of eigen value 2.***

Putting $\lambda = 2$ in $(\lambda I - X) = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -2 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_3 \to R_3 + R_1]$$

$$\Rightarrow \begin{bmatrix} x+y \\ -2x-y-2z \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x+y=0, \ -2x-y-2z=0 \qquad \Rightarrow y=-x, y=-2x-2z$

$\Rightarrow \quad x=-2z, y=2z$

$\therefore \quad \alpha_2=(-2,2,1).$

(III) ***Basis of the eigen space of eigen value 3.***

Putting $\lambda = 3$ in $(\lambda I - X) = O$, we get the homogeneous system of equations

$$\begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\textit{Operating } R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -2 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\textit{Operating } R_1 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\textit{Operating } R_3 \to R_3 + 2R_1]$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\textit{Operating } R_1 \to R_1 + R_2 \textit{ and } R_3 \to R_3 + R_2]$$

$$\Rightarrow \begin{bmatrix} -x-z \\ y-2z \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x-z=0,\ y-2z=0 \qquad \Rightarrow x=-z, y=2z$

$\therefore \quad \alpha_3=(-1, 2, 1).$

Example 11. *Let T be a L.T. on R and* $A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ *be the matrix of T w.r.t. the basis*

$B=\{(3, 0, 0), (1, 2, 0), (4, 6, 5)\}$.

Find all the eigen values and the corresponding vectors of T.

Is T diagonalizable or not ? (P.U. 1998, 95)

Sol. We have $A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ and $B=\{(3, 0, 0), (1, 2, 0), (4, 6, 5)\}$.

The eigen values of T are the values of λ s.t.

$$\det\ [\lambda I - A]=0$$

i.e.,
$$\det\begin{bmatrix} \lambda-3 & -1 & -4 \\ 0 & \lambda-2 & -6 \\ 0 & 0 & \lambda-5 \end{bmatrix}=0$$

i.e., $(\lambda-3)(\lambda-2)(\lambda-5)=0$ *i.e.,* $\lambda=2, 3, 5$.

Thus 2, 3, 5 are the required eigen values of T.

(I) *When* $\lambda = 2$.

Here the corresponding eigen vector $\alpha_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is given by

$$[2I - A]\,\alpha_1 = O \quad i.e., \ [A - 2I]\,\alpha_1 = O$$

i.e.,
$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [\textit{Operating } R_3 \to R_3 - 2R_3]$$

$$\Rightarrow \quad \begin{bmatrix} x + y + 4z \\ 0 \\ 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x + y + 4z = 0,\ 3z = 0$

$\Rightarrow$ $z = 0,\ x + y = 0$

$\Rightarrow$ $x = -y,\ z = 0$

$$\Rightarrow \quad \alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} y.$$

Thus there is only one L.I. vector $\alpha_1 = (-1, 1, 0)$, which is an eigen vector of T corresponding to eigen value 2.

(*II*) *When* $\lambda = 3$.

Here the corresponding eigen vector $\alpha_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is given by

$$[3I - A]\,\alpha_2 = O \quad i.e., \ [A - 3I]\,\alpha_2 = O$$

i.e.,
$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} y + 4z \\ -y + 6z \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $y + 4z = 0,\ -y + 6z = 0,\ 2z = 0$

$\Rightarrow$ $y = 0,\ z = 0,$ where x can assume any value.

Thus the eigen vector corresponding to eigen value 3 is $\alpha_2 = (1, 0, 0)$.

(*III*) *When* $\lambda = 5$.

Here the corresponding eigen vector $\alpha_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is given by

$$[5I - A]\,\alpha_3 = O \quad i.e., \ [A - 5I]\,\alpha_3 = O$$

i.e., $\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2x+y+4z \\ -3y+6z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad -2x + y + 4z = 0, -3y + 6z = 0$

$\Rightarrow \quad 2x = y + 4z, y = 2z.$

Thus the eigen vector corresponding to eigen value 5 is $\alpha_3 = (3, 2, 1)$.

It is noted that eigen vectors corresponding to distinct eigen values are L.I.

Thus the corresponding set of L.I. vectors is $\{(-1, 1, 0), (1, 0, 0), (3, 2, 1)\}$

Since there are 3 L.I. eigen vectors.

$\therefore$ By taking basis of R_3, consisting of the eigen vectors, the operator T can be diagonalized and the matrix T is a diagonal matrix w.r.t. this basis and the diagonal elements are eigen values.

Example 12. *Find all eigen values and basis of eigen space of the matrix*

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

Is A diagonalizable ? (G.N.D.U. 1987 S ; P.U. 1985)

Sol. The eigen values of A are the values of λ s.t.

det. $[\lambda I - A] = 0$

i.e., $$\det \begin{bmatrix} \lambda-1 & 3 & -3 \\ -3 & \lambda+5 & -3 \\ -6 & 6 & \lambda-4 \end{bmatrix} = 0$$

i.e., $$\det \begin{bmatrix} \lambda+2 & 0 & -3 \\ \lambda+2 & \lambda+2 & -3 \\ 0 & \lambda+2 & \lambda-4 \end{bmatrix} = 0$$ [*Operating* $C_1 \to C_1 + C_2$, $C_2 \to C_2 + C_3$]

i.e., $$(\lambda+2)^2 \det \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 1 & \lambda-4 \end{bmatrix} = 0$$

$$\Rightarrow (\lambda+2)^2 \det \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & \lambda-4 \end{bmatrix} = 0$$ [*Operating* $R_2 \to R_2 - R_1$]

$\Rightarrow \quad (\lambda+2)^2(\lambda-4) = 0$

$\Rightarrow \lambda = -2, -2, 4.$

Hence the eigen values of A are -2 and 4.

(I) When $\lambda = -2$.

Here the corresponding eigen vector $\alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is given by

$$[-2\mathrm{I} - \mathrm{A}]\,\alpha = \mathrm{O}$$

i.e.,
$$\begin{bmatrix} -2-1 & 3 & -3 \\ -3 & -2+5 & -3 \\ -6 & 6 & -2-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} -3x+3y-3z \\ -3x+3y-3z \\ -6x+6y-6z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow \quad -3x + 3y - 3z = 0, -3x + 3y - 3z = 0, -6x + 6y - 6z = 0$

$\Rightarrow \quad x - y + z = 0 \qquad \Rightarrow x = y - z.$

Thus y and z are free variables.

Here $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} y - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} z$.

Thus $\alpha_1 = (1, 1, 0)$ and $\alpha_2 = (1, 0, -1)$ are L.I. eigen vectors corresponding to eigen value -2.

Since these are L.I., which generate the eigen space of -2,

$\therefore$ every eigen vector corresponding to -2 is linear combination of α_1 and α_2.

(II) When $\lambda = 4$.

Here the corresponding eigen vector $\alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is given by

$$[4\mathrm{I} - \mathrm{A}]\,\alpha = \mathrm{O}$$

i.e.,
$$\begin{bmatrix} 4-1 & 3 & -3 \\ -3 & 4+5 & -3 \\ -6 & 6 & 4-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 3x+3y-3z \\ -3x+9y-3z \\ -6x+6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 3x + 3y - 3z = 0, -3x + 9y - 3z = 0, -6x + 6y = 0$

$\Rightarrow \quad x + y - z = 0 \qquad \ldots(1)$

$-x + 3y - z = 0 \qquad \ldots(2)$

$-x + y = 0 \qquad \ldots(3)$

$(2) - (1) \Rightarrow -2x + 2y = 0 \quad \Rightarrow -x + y = 0$, which is (3).

Thus we have only two equations :

$x + y - z = 0$ and $-x + y = 0$

$\Rightarrow \quad x = y$ and $z = 2y$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y$$

Thus y is a free variable.

Hence any particular non-zero solution *i.e.*, $x = 1, y = 1, z = 2$ generates the solution space.

Thus $\alpha_3 = (1, 1, 2)$ is an eigen vector, which forms a basis of eigen space of 4.

Since A has three linearly independent eigen vectors,

$\therefore$ A is diagonalizable.

Let P be the matrix whose columns are three L.I. vectors

i.e., Let $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

Then $P^{-1} A P = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

The diagonal elements of $P^{-1} AP$ are eigen values of A corresponding to columns of P.

Hence A is diagonalizable. [$\because$ *A has 3 L.I. eigen vectors*]

Example 13. *Suppose λ is an eigen value of an operator T. Show that $f(\lambda)$ is an eigen value of $f(T)$.*

(G.N.D.U. 1992 S, 88 S)

Sol. Let $v \in V$ be an eigen vector of T associated with the eigen value λ.

Then $v \neq 0$ and $Tv = \lambda v$ ...(1)

To prove. *$T^m(v) = \lambda^m v$ for all + ve integral m* ...(2)

For $m = 1$, the result is true by (1).

Let the result be true for a + ve integer k

i.e., $T^k(v) = \lambda^k v$ ...(3)

Then $T^{k+1}(v) = (T^k T)(v) = T^k(Tv)$ [*Using (1)*]

$= T^k(\lambda v) = \lambda(T^k(v))$ [$\because$ *T^k is linear*]

$= \lambda(\lambda^k v)$ [*Using (3)*]

$= (\lambda\lambda^k) v = \lambda^{k+1} v$

Hence, by induction, $T^m(v) = \lambda^m v$ for all integral m.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m$

be any polynomial over T,

then $f(T) = a_0 I + a_1 T + a_2 T^2 + \ldots\ldots + a_m T^m$

Now $(f(T))(v) = (a_0 I + a_1 T + a_2 T^2 + \ldots\ldots + a_m T^m)(v)$

$= (a_0 I)(v) + (a_1 T)(v) + (a_2 T^2)(v) + \ldots\ldots + (a_m T^m)(v)$

$= a_0(Iv) + a_1(Tv) + a_2(T^2 v) + \ldots\ldots + a_m(T^m v)$

Then there exists an invertible matrix P such that

$$B = P^{-1} AP \qquad ...(1) \text{ [By def.]}$$

Also $tI = P^{-1} tIP$ ...(2) $[\because P^{-1} tIP = t P^{-1}IP = t P^{-1} P = tI]$

Now $|tI - B| = |tI - P^{-1} AP|$ $[\because \text{of}(1)]$

$= |P^{-1} tIP - P^{-1} AP|$ $[\because \text{of}(2)]$

$= |P^{-1} (tI - A) P|$

$= |P^{-1}| \, |tI - A| \, |P| = |P^{-1}| \, |P| \, |tI - A|$ ...(3)

Since determinants are scalars, and $|P^{-1}| \, |P| = |P^{-1} P| = |I| = 1$

$\therefore$ from (3), $|tI - B| = |tI - A|$

$\Rightarrow$ A and B have same characteristic polynomial.

Hence the result.

6. Characteristic Polynomial of A Linear Operator

Def. *Let $T : V \to V$ be a linear operator on a vector space V with finite dimension over F, then characteristic polynomial Δ (t) of T is said to be the* **characteristic polynomial** *of any matrix representation of T.*

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Theorem I. *If T be a linear operator on a finite dimensional space V and λ be a scalar, then λ is an eigen value of T iff det $(T - \lambda I) = 0$ (i.e., iff $\lambda I - T$ is singular) i.e., iff λ is a root of characteristic polynomial of T.* (P.U. 1985)

Proof. λ is an eigen value of T

iff there exists a non-zero vector $v \in V$ such that $Tv = \lambda v$

iff $(T - \lambda I) v = 0$ iff $T - \lambda I$ is a singular operator

iff det. $(T - \lambda I) = 0$ iff det. $(\lambda I - T) = 0$

iff λ satisfies the polynomial $|T - \lambda I|$

iff λ is a root of characteristic polynomial of T.

Remark. If λ is an eigen value of T, *i.e.*, if λ is a root of characteristic polynomial of T, then det $(T - \lambda I) = 0$.

Further degree of characteristic polynomial Δ (t) of T = dim V.

$\therefore$ The no. of eigen values of T can't be more than the dim V.

Theorem II. *Let T be a linear operator on a finite dimensional space V and $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigen values of T. Then T is diagonalizable iff the characteristic polynomial for T is*

$$\Delta(t) = (t - \lambda_1)^{d_1} (t - \lambda_2)^{d_2} ......(t - \lambda_k)^{d_k},$$

where $d_1 + d_2 + ...+ d_k = \dim V$.

Proof. Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ be distinct eigen values of T and dim V = n. T is diagonalizable iff there exists an ordered basis B in which T is represented by a diagonal matrix which has for its diagonal entries the scalars $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ (each being repeated many times). If c_i is repeated d_i times, then

$$A = [T : n] = \begin{bmatrix} \lambda_1 I_1 & 0 & \ldots\ldots 0 \\ 0 & \lambda_2 I_2 & \ldots\ldots 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots \lambda_k I_k \end{bmatrix}, \text{ where } I_j \text{ is } d_j \times d_j \text{ identity matrix.}$$

Now T is diagonalizable iff there exists an ordered basis B such that characteristic of T is

$$\Delta(t) = \det(tI - A)$$

$$= \begin{bmatrix} (t-\lambda_1) I_1 & 0 & \ldots\ldots 0 \\ 0 & (t-\lambda_2) I_2 & \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \\ 0 & 0 & \ldots\ldots (t-\lambda_k) I_k \end{bmatrix},$$

where degree of characteristic polynomial = dim V.

But degree of characteristic polynomial = $d_1 + d_2 + \ldots\ldots + d_k$

$\therefore \quad d_1 + d_2 + \ldots\ldots + d_k = n$

$\therefore \quad \Delta(t) = (t-\lambda_1)^{d_1} (t-\lambda_2)^{d_2} \ldots\ldots (t-\lambda_k)^{d_k},$

where $d_1 + d_2 + \ldots\ldots + d_k = n$.

Hence T is diagonalizable iff characteristic polynomial of T is

$$\Delta(t) = (t-\lambda_1)^{d_1} (t-\lambda_2)^{d_2} \ldots\ldots (t-\lambda_k)^{d_k},$$

where $d_1 + d_2 + \ldots\ldots + d_k = n$.

Theorem III. Cayley — Hamilton Theorem

Let T be a linear operator on a finite dimensional vector space V over F. If $\Delta(t)$ is the characteristic polynomial for T, then $\Delta(T) = 0$ (i.e., T satisfies its characteristic polynomial).

(Pbi. U. 1997, 96 ; G.N.D.U. 1996, 93, 90, 89)

Proof. Let the ordered basis of V be $\{v_1, v_2, \ldots\ldots, v_n\}$.

Let A be the matrix which represents T in $\{v_1, v_2, \ldots\ldots, v_n\}$.

$\therefore \quad T(v_{ij}) = \Sigma A_{ij} v, 1 \le i \le n$ ...(1),

where $A = [A_{ij}]_{n \times n}$

The equivalent form of the equation is

$\Sigma(\delta_{ij} ; T - A_{ji} I) v_i = 0, 0 \le i \le n$ ...(2), where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

We denote B as the matrix of ordered $n \times n$ with entries

$$B_{ij} = \delta_{ij} T - A_{ij} I \qquad \ldots(3)$$

$\therefore$ From (2), $\Sigma B_{ij} v_i = 0, 0 \leq i \leq n$

When $n = 2$.

Here $B = \begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{12} I & T - A_{22} I \end{bmatrix}$

$$\therefore \det B = \det \begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{12} I & T - A_{22} I \end{bmatrix}$$

$$= (T - A_{11}I)(T - A_{22}I) - A_{12} A_{21} I \qquad [\because II = I]$$

$$= T^2 - (A_{11} + A_{22}) T + (A_{11} A_{22} - A_{12} A_{21}) I$$

$$= \Delta(T).$$

To prove. $\Delta(t)$ is the characteristic polynomial of T.

When $n = 2$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$\therefore$ Characteristic polynomial of T is

$$\Delta(t) = \det(tI - A) = \det \begin{bmatrix} t - A_{11} & -A_{12} \\ -A_{21} & t - A_{22} \end{bmatrix}$$

$$= (t - A_{11})(t - A_{22}) - A_{12} A_{21} = t^2 - (A_{11} + A_{22}) t + (A_{11} A_{22} - A_{21} A_{12}).$$

Thus $\Delta(t)$ is the characteristic polynomial for T.

When $n > 2$.

Here also det. $B = \Delta(T)$.

To prove . $\Delta(T) = 0$.

For $\Delta(T)$ to be zero operator, it is necessary and sufficient that $(\det B)\, \alpha_k = 0$ for $k = 1, 2, \ldots\ldots, n$.

$[\because \det B = \Delta(T)]$

By def., from (3), the vectors $v_1, v_2, \ldots\ldots, v_n$ satisfy

$$\sum_{j=1}^{n} B_{ij} v_j = 0, \; 1 \leq i \leq n.$$

When $n = 2$, (3) takes the form

$$\begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{21} I & T - A_{22} I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \ldots(4)$$

Here adj B = $\begin{bmatrix} T - A_{22}I & A_{21}I \\ A_{12}I & T - A_{11}I \end{bmatrix}$

Also (adj B) B = B (adj B) = (det B) I

$\therefore$ (adj B) B = $\begin{bmatrix} \det B & 0 \\ \det B & 0 \end{bmatrix}$

$\therefore$ (det B) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = (adj. B B) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = adj B $\left[B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right]$ = (adj B) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [*Using (4)*]

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Let B = adj B, in general, then from (3),

$$\sum_{j=1}^{n} \bar{B}_{ki} B_{ij} \alpha_j = 0 \; \forall \; k, i$$ [$\because$ $\bar{B}_{ki}$ *is* (k, i)*th entry of* $\bar{B}$]

Summing on i, $0 = \sum_{j=1}^{n} \sum_{j=1}^{n} \bar{B}_{ki} B_{ij} \alpha_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \bar{B}_{ki} B_{ij} \right) \alpha_j$ ...(5)

But $\bar{B}$ B = det I, $\therefore \sum_{j=1}^{n} \bar{B}_{ki} B_{ij} = \delta_{kj} \det B$,

where $\delta_{ij} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$

$\therefore$ from (5), $0 = \sum_{j=1}^{n} \delta_{kj}$ (det B) α_j, $0 \le k \le n =$ (det B) α_k

Thus (det B) $\alpha_k = 0$ for $0 \le k \le n$

$\therefore$ $(\Delta \, T) \alpha_k = 0$ for $0 \le k \le n$

$\Rightarrow$ $\Delta(T) = 0$.

Hence the theorem.

7. Minimal Polynomials

Definitions :

(*i*) **Minimal Polynomial of a Matrix.** *Let A be a n* $\times$ *n matrix over F. Then the monic polynomial m(t) over F is said to be the minimal polynomial of A if m (t) is of lowest degree such that m (A) = O i.e., A is a zero of m (t).*

(*ii*) **Minimal Polynomial of an Operator.** *Let T be an operator of a finite dimensional vector space V over F. Then the polynomial m (t) is said to be the minimal polynomial of T if m (t) is of lowest degree with leading co-efficient 1 s.t. T is a zero of m (t) i.e., m (T) = O.*

THEOREMS

Theorem I. *Let T be a linear operator on a finite dimensional vector space V over F and if T is represented in some ordered basis by the matrix A, then T and A have the same minimal polynomial.*

Proof. Let $f(t) = a_0 + a_1 t + \ldots\ldots + a_{n-1} t^{n-1} + a_n t^n$ be any polynomial over F.

Firstly, to show that if the matrix representation of T w.r.t. some ordered basis of A, then matrix representation of f(T) is f(A).

We have $[T] = A$. *[Given]*

Now $f(T) = a_0 I + a_1 T + ...+ a_{n-1} T^{n-1} + a_n T^n$.

Let $[f(T)]$ be the matrix representation of $f(T)$.

$$[f(T)] = [a_0 I + a_1 T + ... + a_{n-1} T^{n-1} + a_n T^n]$$

$$= [a_0 I] + [a_1 T] + ...+ [a_{n-1} T^{n-1}] + [a_n T^n] \qquad [\textit{Using } [T_1 + T_2] = [T_1] + [T_2]]$$

$$= a_0 [I] + a_1 [T] + ...+ a_{n-1} [T^{n-1}] + a_n [T^n] \qquad [\textit{Using } [\alpha T] = \alpha[T]]$$

$$= a_0 I + a_1 A + ...+ a_{n-1} A^{n-1} + a_n A^n$$

$$= f(A).$$

Let $m(t)$ be the minimal polynomial of T.

Then $m(A)$ is the matrix representation of $m(T)$.

$\therefore \quad m(T) = O$ iff $m(A) = O$.

Thus $m(t)$ is the minimal polynomial of T iff $m(t)$ is the minimal polynomial of A.

Hence T and A have the same minimal polynomial.

Theorem II. *The minimal polynomial $m(t)$ of an operator T divides every polynomial which has T as a zero.*

In particular, *$m(t)$ divides characteristic polynomial of T.*

Proof. Since $f(t)$ is any polynomial s.t. $f(T) = O$.

By division algorithm, there exists polynomials $q(t)$ and $r(t)$ *s.t.*

$$f(t) = m(t)\, q(t) + r(t) \qquad ...(1),$$

where either $r(t) = 0$ or $\deg(r(t)) < \deg(m(t))$.

Putting $t = T, f(T) = m(T)\, q(T) + r(T)$

$\Rightarrow \quad O = O + r(T) \qquad [\because\ f(T) = O = m(T)]$

$\Rightarrow \quad r(T) = O.$

If $r(t) \neq 0$, then $r(t)$ is a polynomial of degree less than that of $m(t)$, which has T as a zero.

This leads to contradiction. *[By def. of minimal polynomial]*

Thus $r(t) = 0$ is only possible.

$\therefore$ From (1), $f(t) = m(t)\, q(t)$.

Hence $m(t) \mid f(t)$.

Theorem III. *Suppose $T\alpha = c\alpha$. If c is an eigen value of T, then $f(c)$ is an eigen value of $f(T)$.*

Or

If $f(t)$ is any polynomial, then $f(T)\alpha = f(c)\alpha$.

Proof. Let $f(t) = a_0 + a_1 t + ...+ a_{n-1} t^{n-1} + a_n t^n$

$\therefore \quad f(T) = a_0 I + a_1 T + ...+ a_{n-1} T^{n-1} + a_n T^n$

$$\therefore \quad f(T)\alpha = (a_0 I + a_1 T + ...+ a_{n-1} T^{n-1} + a_n T^n)\alpha$$

$$= (a_0 I)\alpha + (a_1 T)\alpha + ...+ (a_{n-1} T^{n-1})\alpha + (a_n T^n)\alpha$$

$$= a_0 I\alpha + a_1 T\alpha + ...+ a_{n-1} (T^{n-1})\alpha + a_n (T^n)\alpha \qquad ...(1)$$

But $\quad T\alpha = c\alpha$ *[Given]*

$\therefore \quad (T^2)\alpha = (TT)\alpha = T(T\alpha) = T(c\alpha) = c(T\alpha) = c(c\alpha) = c^2\alpha$

Assuming that $T^{n-1}\alpha = c^{n-1}\alpha$

$\therefore \quad (T^n)\alpha = (T\,T^{n-1})\alpha = T\,(T^{n-1}\alpha) = T\,(c^{n-1}\alpha)$ *[Assumption]*

$= c^{n-1}(T\alpha) = c^{n-1}(c\alpha) = c^n\alpha.$

Thus $\quad T^n\alpha = c^n\alpha\ \forall\ n.$

Putting in (1), $f(T)\alpha = a_0\alpha + a_1 c\alpha + \ldots + a_{n-1}c^{n-1}\alpha + a_n c^n\alpha$

$= (a_0 + a_1 c + \ldots\ldots + a_{n-1}c^{n-1} + a_n c^n)\alpha$

Hence $\quad f(T)\alpha = f(c)\alpha.$

Theorem IV. *The characteristic and minimal polynomials for an operator T (or a matrix A) have same irreducible factors.*

Or

Let T be a linear operator on n-dimensional vector space V (or A be n × n matrix). Then the characteristic and minimal polynomials for T (or A) have same roots except for multiplicities.

Proof. Let $m(t)$ be the minimal polynomial for T.

Let c be a scalar.

To prove. $m(c) = 0$ iff c is an eigen value of T.

Suppose $m(c) = 0$.

Then $m(t) = (t - c)\,q(t)$ ...(1), where $q(t)$ is a polynomial.

Further as $\deg(q(t)) < \deg(m(t))$,

$\therefore \quad q(T) \neq O$ *[By def. of minimal polynomial m (t)]*

Let us select a vector $\beta \in V$ s.t. $q(T)\beta \neq 0$.

Let $\alpha = q(T)\beta$. Then $m(T)\beta = 0$ $[\because\ m(T) = 0]$

$\Rightarrow \quad (T - cI)\,q(T)\beta = 0$ $[\because$ *From (1),* $m(T) = (T - cI)\,q(T)]$

$(T - cI)\beta = 0$

$\Rightarrow \quad c$ is a given value of T $\quad \Rightarrow c$ is a root of characteristic equation of T.

Conversely :

Here c is an eigen value of T, *i.e.,* c is a root characteristic equation of T.

Then there exists a non-zero vector $\alpha \in V$ *s.t.* $T\alpha = c\alpha$.

Since $m(T)\alpha = m(c)\alpha$ *[Th. III]*

Now $m(T) = O$ and $\alpha \neq 0$. $\quad \therefore \quad m(c) = 0$

$\Rightarrow \quad c$ is a root of the minimal polynomial $m(t)$.

Hence the result.

Theorem V. *λ (a scalar) is an eigen value for the operator T on V iff λ is a root of the minimal polynomial of T.*

Proof. We know that λ is an eigen value for T iff λ is a root of the characteristic polynomial $\Delta(t)$ of T.

But the characteristic polynomial and minimal polynomial have same roots. *[Th. IV]*

Hence λ is an eigen value for T iff λ is a root of the minimal polynomial $m(t)$ of T.

SOLVED EXAMPLES

Example 1. *Find the characteristic polynomial of the matrix*

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}.$$

Example 4. *Find the polynomial whose one root is*

(*i*) $A = \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix}$ (*ii*) $B = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & 1 \end{bmatrix}$.

Sol. (*i*) We have $A = \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix}$.

The required polynomial of A is

$$\Delta(t) = |tI - A| = \det \begin{bmatrix} t-3 & 7 \\ -4 & t-5 \end{bmatrix}$$

$$= (t-3)(t-5) - (-28) = t^2 - 8t + 15 + 28 = t^2 - 8t + 43.$$

(*ii*) We have $B = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & 1 \end{bmatrix}$.

The required polynomial of B is

$$\Delta(t) = |tI - B| = \det \begin{bmatrix} t-2 & -3 & 2 \\ 0 & t-5 & -4 \\ -1 & 0 & t-1 \end{bmatrix}$$

$$= (t-2)\det \begin{bmatrix} t-5 & -4 \\ 0 & t-1 \end{bmatrix} - 1 \det \begin{bmatrix} -3 & 2 \\ t-5 & -4 \end{bmatrix} \qquad \text{[Expanding by } C_1\text{]}$$

$$= (t-2)(t-5)(t-1) - [12 - 2(t-5)] = (t^2 - 7t + 10)(t-1) - 12 + 2t - 10$$

$$= t^3 - 7t^2 + 10t - t^2 + 7t - 10 - 12 + 2t - 10$$

$$= t^3 - 8t^2 + 19t - 32.$$

Example 5. *Find the characteristic polynomial of the matrix*

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Sol. Characteristic polynomial of A is $\Delta(t) = |It - A|$

$$= \det \begin{bmatrix} t-3 & -1 & 0 & 0 & 0 \\ 0 & t-3 & 0 & 0 & 0 \\ 0 & 0 & t-3 & -1 & 0 \\ 0 & 0 & 0 & t-3 & -1 \\ 0 & 0 & 0 & 0 & t-3 \end{bmatrix}$$

$$= (t-3)\det\begin{bmatrix} t-3 & 0 & 0 & 0 \\ 0 & t-3 & -1 & 0 \\ 0 & 0 & t-3 & -1 \\ 0 & 0 & 0 & t-3 \end{bmatrix} \qquad [\textit{Expanding by } C_1]$$

$$= (t-3)^2\det\begin{bmatrix} t-3 & -1 & 0 \\ 0 & t-3 & -1 \\ 0 & 0 & t-3 \end{bmatrix} \qquad [\textit{Expanding by } C_1]$$

$$= (t-3)^3\det\begin{bmatrix} t-3 & -1 \\ 0 & t-3 \end{bmatrix} \qquad [\textit{Expanding by } C_1]$$

$$= (t-3)^3\,[(t-3)^2 - 0] = (t-3)^5.$$

Example 6. *Let T be a linear operator on R^4 and the matrix representation of T w.r.t some ordered basis be*

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Find the minimal polynomial $m(t)$ of T. (G.N.D.U. 1995 S, 93 ; Pbi. U. 1986)

Sol. The characteristic polynomial of T is

$$\Delta(t) = \det[tI - A] = \det\begin{bmatrix} t-2 & 1 & 0 & 0 \\ 0 & t-2 & 0 & 0 \\ 0 & 0 & t-2 & 0 \\ 0 & 0 & 0 & t-5 \end{bmatrix}$$

$$= (t-2)^3(t-5)$$

Thus $\quad \Delta(t) = (t-2)^3(t-5)$.

But characteristic polynomial and minimal polynomial have same irreducible factors.

$\therefore$ both $(t-2)$ and $(t-5)$ are factors of $m(t)$.

Since $m(t)$ divides $\Delta(t)$,

$\therefore$ $m(t)$ is one of the following polynomials

$m_1(t) = (t-2)(t-5)$, $m_2(t) = (t-2)^2(t-5)$, $m_3(t) = (t-2)^3(t-5)$.

Now $m_1(A) = (A-2I)(A-5I)$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq O,$$

$$m_2(A) = (A-2I)^2(A-5I) = (A-2I)[(A-2I)(A-5I)]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O,$$

$m_3(A) = O$ [*By Cayley-Hamilton Theorem*]

Thus minimal polynomial of A is

$$m(t) = (t-2)^2(t-5).$$

But minimal polynomial of A and T are same.

$\therefore$ the minimal polynomal for T is.

$$m(t) = (t-2)^2(t-5).$$

Example 7. *Let T be a linear operator on T^2 which is represented in the standard basis by the matrix*

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

Find the characteristic and minimal polynomial for T.

Sol. Here $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$

Characteristic polynomial of A is

$$|tI - A| = \begin{bmatrix} t-5 & 6 & 6 \\ 1 & t-4 & -2 \\ -3 & 6 & t+4 \end{bmatrix} = \begin{bmatrix} t-5 & 0 & 6 \\ 1 & t-2 & -2 \\ -3 & -(t-2) & t+4 \end{bmatrix} \quad [\textit{Operating } C_2 \to C_2 - C_3]$$

$$= (t-2)\begin{bmatrix} t-5 & 0 & 6 \\ 1 & 1 & -2 \\ -3 & -1 & t+4 \end{bmatrix}$$

$$= (t-2)\left[(t-5)(t+4-2) + 6(-1+3)\right] = (t-2)[(t-5)(t+2) + 12]$$

$$= (t-2)(t^2-3t+2) = (t-2)(t-2)(t-1)$$

$$= (t-2)^2(t-1).$$

Hence $\Delta(t) = (t-2)^2(t-1)$ is the characteristic polynomial.

Since the characteristic polynomial and minimal polynomial have the same irreducible factors,

$\therefore$ $(t-1)$ and $(t-2)$ are factors of $m(t)$

$\therefore$ minimal polynomial $m(t)$ is one of the following :

$$m_1(t) = (t-2)(t-1) \text{ and } m_2(t) = (t-2)^2(t-1).$$

Now $$A^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0-0 & 0+0-1 & -2+0-3 \\ 10+5+0 & 0+1+0 & -5+0+0 \\ 0+5+0 & 0+1+3 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$$

$\therefore$ From (1), $$A^{-1} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} + \begin{bmatrix} -12 & 0 & 6 \\ -30 & -6 & 0 \\ 0 & -6 & -18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12+11 & -1+0+0 & -5+6+0 \\ 15-30+0 & 1-6+11 & -5+0+0 \\ 5+0+0 & 4-6+0 & 9-18+11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

Example 12. *Find the characteristic equation of the matrix*

(*i*) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (G.N.D.U. 1998 ; P.U. 1992)

(*ii*) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ (P.U. 1995)

(*iii*) $A = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (P.U. 1995, 91)

and using Cayley-Hamilton Theorem find A^{-1}.

Sol. (*i*) (*a*) Characteristic polynomial of A is

$$\Delta(t) = |tI - A|$$

$$= \det\begin{bmatrix} t-2 & 1 & -1 \\ 1 & t-2 & 1 \\ -1 & 1 & t-2 \end{bmatrix}$$

$$= \det\begin{bmatrix} t-1 & t-1 & 0 \\ 1 & t-2 & 1 \\ 0 & t-1 & t-1 \end{bmatrix}$$ [*Operating* $R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$]

$$= \begin{bmatrix} -\frac{4}{9}+\frac{1}{3}+\frac{1}{9} & -\frac{1}{3}+\frac{2}{3}+0 & 0+\frac{1}{3}+0 \\ \frac{1}{3}+0+0 & -\frac{2}{9}+\frac{1}{3}+\frac{1}{9} & \frac{2}{9}-\frac{1}{3}+0 \\ -\frac{2}{3}+1+0 & -\frac{4}{9}-\frac{1}{3}+0 & -\frac{5}{9}+\frac{1}{3}+\frac{1}{9} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{3} & -\frac{7}{9} & -\frac{1}{9} \end{bmatrix}.$$

(*iii*) We have $A = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*a*) Characteristics polynomial of A is

$$A(t) = |tI - A| = \det \begin{bmatrix} t-1 & -\sqrt{2} & 0 \\ -\sqrt{2} & t+1 & 0 \\ 0 & 0 & t-1 \end{bmatrix} = (t-1)\,[(t^2-1)-2] = (t-1)(t^2-3) = t^3 - t^2 - 3t + 3.$$

(*b*) By Cayley Hamilton Theorem,

$A^3 - A^2 - 3A + 3I = O$

so that $\quad I = -\frac{1}{3}(A^3 - A^2 - 3A) \quad \Rightarrow \quad I = -\frac{1}{3} A(A^2 - A - 3I)$

$$\Rightarrow \quad A^{-1} = -\frac{1}{3}A^2 + \frac{1}{3}A + \frac{1}{3} I \qquad \ldots(1)$$

Now $A^2 = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+2+0 & \sqrt{2}-\sqrt{2}+0 & 0+0+0 \\ \sqrt{2}-\sqrt{2}+0 & 2+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\therefore$ From (1), $A^{-1} = -\frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Now $\quad A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$\therefore$ From (1),

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{2}{3} & \frac{5}{9} & -\frac{5}{9} \\ \frac{5}{9} & -\frac{2}{3} & \frac{5}{9} \\ -\frac{5}{9} & \frac{5}{9} & -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9}+\frac{2}{3}-\frac{2}{3} & 0-\frac{1}{3}+\frac{5}{9} & 0+\frac{1}{3}-\frac{5}{9} \\ 0-\frac{1}{3}+\frac{5}{9} & \frac{1}{9}+\frac{2}{3}-\frac{2}{3} & 0-\frac{1}{3}+\frac{5}{9} \\ 0+\frac{1}{3}-\frac{5}{9} & 0-\frac{1}{3}+\frac{5}{9} & \frac{1}{9}+\frac{2}{3}-\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} & -\frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

(*b*) **Exactly similar to part (*a*).** $\left[\textbf{Ans. } \frac{1}{3}\begin{pmatrix} 3 & -6 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}\right]$

(*c*) **Exactly similar to part (*a*).** $\left[\textbf{Ans. } \frac{1}{4}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\right]$

Example 14. (*a*) *Let T be a linear operator on* R^3 *defined by*

$$T(x, y, z) = (2x - y, x + y + z, 2z).$$

Find the characteristic and minimal polynomials for T and verify Cayley-Hamilton Theorem.

(G.N.D.U. 1985 S)

(*b*) *Let T be a linear operator on* R^4 *defined by*

$$T(x, y, z, t) = (x + y, 2y, z + t, 2z + 4t).$$

Find the characteristic and minimal polynomials for T and verify Cayley-Hamilton Theorem.

(G.N.D.U. 1985 ; Pbi. U. 1985)

Sol. (*a*) The standard basis for R^3 is

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

By def., $T(x, y, z) = (2x - y, x + y + z, 2z)$ ...(1)

$T(1, 0, 0) = (2 - 0, 1 + 0 + 0, 0) = (2, 1, 0)$

$T(0, 1, 0) = (0 - 1, 0 + 1 + 0, 0) = (-1, 1, 0)$

and $T(0, 0, 1) = (0 - 0, 0 + 0 + 1, 2) = (0, 1, 2)$

$\therefore$ $T(1, 0, 0) = 2(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1)$ ...(2)

$T(0, 1, 0) = -1(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1)$ ...(3)

and $T(0, 0, 1) = 0(1, 0, 0) + 1(0, 1, 0) + 2(0, 0, 1)$ ...(4)

$\therefore$ Matrix of T w.r.t. (1) is

$$[T] = A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}^t = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(*i*) Characteristic polynomial of T is

$$\Delta(t) = |It - A| = \begin{vmatrix} t-2 & 1 & 0 \\ -1 & t-1 & -1 \\ 0 & 0 & t-2 \end{vmatrix} = (t-2)[(t-1)(t-2)+1] = (t-2)(t^2 - 3t + 3)$$

$\therefore$ $\Delta(t) = t^3 - 5t^2 + 9t - 6.$

(*ii*) The characteristic polynomial and minimal polynomial have the same irreducible factors, but the characteristic polynomial is

$$\Delta(t) = (t-2)(t^2 - 3t + 3),$$

which is the product of two irreducible factors, one being linear and the other a quadratic.

$\therefore$ The minimal polynomial $m(t)$ of T is the same as the characteristic polynomial $\Delta(t)$.

Hence the minimal polynomial is $m(t) = t^3 - 5t^2 + 9t - 6$.

(*iii*) **Verification of Cayley-Hamilton Theorem**

We have $T(x, y, z) = (2x - y, x + y + z, 2z)$

$$\begin{aligned} T^2(x, y, z) &= T[T(x, y, z)] = T(2x - y, x + y + z, 2z) \\ &= (4x - 2y - x - y - z, 2x - y + x + y + z + 2z, 4z) \\ &= (3x - 3y - z, 3x + 3z, 4z) \end{aligned}$$

$$\begin{aligned} T^3(x, y, z) &= T[T^2(x, y, z)] = T(3x - 3y - z, 3x + 3z, 4z) \\ &= (6x - 6y - 2z - 3x - 3z, 3x - 3y - z + 3x + 3z + 4z, 8z) \\ &= (3x - 6y - 5z, 6x - 3y + 6z, 8z) \end{aligned}$$

$\Delta(T) = T^3 - 5T^3 + 9T - 6I.$

$$\begin{aligned} \therefore\quad [\Delta(T)](x, y, z) &= (T^3 - 5T^2 + 9T - 6I)(x, y, z) \\ &= (3x - 6y - 5z, 6x - 3y + 6z, 8z) - 5(3x - 3y - z, 3x + 3z, 4z) \\ &\qquad + 9(2x - y, x + y + z, 2z) - 6(x, y, z) \\ &= (3x - 6y - 5z - 15x + 15y + 5z + 18x - 9y - 6x, 6x - 3y + 6z - 15x - 15z + 9x \\ &\qquad + 9y + 9z - 6y, 8z - 20z + 18z - 6z) \\ &= (0, 0, 0) = O(x, y, z) \end{aligned}$$

$\therefore$ $[\Delta(T)](x, y, z) = O(x, y, z)\ \forall (x, y, z) \in R^3$

$\Rightarrow$ $\Delta(T) = O.$

Hence the verification.

(*b*) **Exactly similar to part (*a*).** [**Ans.** $\Delta(t) = (t-3)(t-2)^2(t-1)$; $m(t) = (t-3)(t-2)(t-1)$]

Example 15. *Find the characteristic polynomial of the linear transformation $T : R^3 \to R^3$ defined by*

$$T(a, b, c) = (a + b, b + c, c + a)$$

and verify Cayley-Hamilton Theorem. (G.N.D.U. 1992 S)

Sol. The standard basis for R^3 is

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

By def., $T(a, b, c) = (a + b, b + c, c + a)$ ...(1)

$$T(1, 0, 0) = (1 + 0, 0 + 0, 0 + 1) = (1, 0, 1)$$
$$T(0, 1, 0) = (0 + 1, 1 + 0, 0 + 0) = (1, 1, 0)$$
$$T(0, 0, 1) = (0 + 0, 0 + 1, 1 + 0) = (0, 1, 1)$$

$\therefore$ $T(1, 0, 0) = 1 . (1, 0, 0) + 0 . (0, 1, 0) + 1 . (0, 0, 1)$

$T(0, 1, 0) = 1 . (1, 0, 0) + 1 . (0, 1, 0) + 0 . (0, 0, 1)$

$T(0, 0, 1) = 0 . (1, 0, 0) + 1 . (0, 1, 0) + 1 . (0, 0, 1)$

Matrix of T w.r.t. (*i*) is

$$[T] = A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(*i*) Characteristic polynomial of T is

$$\Delta(t) = |It - A|$$

$$= \begin{vmatrix} t-1 & -1 & 0 \\ 0 & t-1 & -1 \\ -1 & 0 & t-1 \end{vmatrix} = (t-1)^3 - (1) = t^3 - 3t^2 + 3t - 2.$$

(*ii*) **Verification of Cayley-Hamilton Theorem**

$$T(a, b, c) = (a + b, b + c, c + a)$$

$$T^2(a, b, c) = T[T(a, b, c)] = T(a + b, b + c, c + a)$$
$$= (a + b + b + c, b + c + c + a, c + a + a + b)$$
$$= (a + 2b + c, a + b + 2c, 2a + b + c)$$

$$T^3(a, b, c) = T[T^2(a, b, c)] = T(a + 2b + c, a + b + 2c, 2a + b + c)$$
$$= (a + 2b + c + a + b + 2c, a + b + 2c + 2a + b + c, 2a + b + c + a + 2b + c)$$
$$= (2a + 3b + 3c, 3a + 2b + 3c, 3a + 3b + 2c)$$

$$\Delta(T) = T^3 - 3T^2 + 3T - 2I$$

$\therefore$ $[\Delta(T)](a, b, c) = (T^3 - 3T^2 + 3T - 2I)(a, b, c)$

$$= (2a+3b+3c, 3a+2b+3c, 3a+3b+2c) - 3(a + 2b + c, a + b + 2c, 2a + b + c) + 3(a + b, b + c, c + a) - 2(a, b, c)$$

$$= (2a + 3b + 3c - 3a - 6b - 3c + 3a + 3b - 2a, 3a + 2b + 3c - 3a - 3b - 6c + 3b + 3c - 2b, 3a + 3b + 2c - 6a - 3b - 3c + 3c + 3a - 2c)$$

$$= (0, 0, 0) = O(a, b, c)$$

$\therefore$ $[\Delta(T)](a, b, c) = O(a, b, c)\ \forall (a, b, c) \in R^3$

$\Rightarrow$ $\Delta(T) = O.$

Hence the verification.

Example 16. *Find the characteristic polynomial of the transformation* $T: R^3 \to R^3$ *defined by*

$$T(a, b, c) = (a + 2b + c, b - c, 3a - b + c)$$

and use the Cayley-Hamilton Theorem to find T^{-1}. (G.N.D.U. 1992)

Sol. The standard basis for R^3 is

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

By def., $T(a, b, c) = (a + 2b + c, b - c, 3a - b + c)$ ...(1)

$$T(1, 0, 0) = (1 + 2(0) + 0, 0 - 0, 3(1) - 0 + 0) = (1, 0, 3)$$

$$T(0, 1, 0) = (0 + 2(1) + 0, 1 - 0, 3(0) - 1 + 0) = (2, 1, -1)$$

and $T(0, 0, 1) = (0 + 2(0) + 1, 0 - 1, 3(0) - 0 + 1) = (1, -1, 1)$

$\therefore$ $T(1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 3(0, 0, 1)$

$$T(0, 1, 0) = 2(1, 0, 0) + 1(0, 1, 0) - 1.(0, 0, 1)$$

and $T(0, 0, 1) = 1(1, 0, 0) - 1.(0, 1, 0) + 1.(0, 0, 1)$

$\therefore$ Matrix of T w.r.t. (*i*) is

$$[T] = A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}.$$

(*i*) Characteristic polynomial of T is

$$\Delta(t) = |It - A|$$

$$= \begin{bmatrix} t-1 & -2 & -1 \\ 0 & t-1 & 1 \\ -3 & 1 & t-1 \end{bmatrix} = (t-1)\left((t-1)^2 - 1\right) - 3(-2 + t - 1)$$

$$= (t-1)^3 - (t-1) - 3(t-3) = t^3 - 3t^2 + 3t - 1 - t + 1 - 3t + 9$$

$$= t^3 - 3t^2 - t + 9.$$

(*ii*) **Exactly similar to Ex. 12 (a) (*ii*).**

Example 17. *Let V be the vector space of functions which have* $(\sin\theta, \cos\theta)$ *as a basis. Let D be the differential operator on V. Then find the characteristic polynomial for D and verify Cayley-Hamilton Theorem.*

Sol. We have $B = \{\sin\theta, \cos\theta\}$

$\therefore$ $D(\sin\theta) = \cos\theta = 0 . \sin\theta + 1 . \cos\theta$

and $D(\cos\theta) = -\sin\theta = -1 . \sin\theta + 0\cos\theta$

$\therefore$ $[D] = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\therefore$ The characteristic polynomial for D is

$$\det[tI - A] = \det\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix} = t^2 + 1.$$

Verification of Cayley-Hamilton Theorem

$$D^2 + I = O.$$

Now $(D^2 + 1)\sin\theta = D^2(\sin\theta) + I(\sin\theta) = D(D\sin\theta) + \sin\theta = D(\cos\theta) + \sin\theta$

$$= -\sin\theta + \sin\theta = 0$$

and $(D^2 + 1)\cos\theta = D^2(\cos\theta) + 1(\cos\theta) = D(D\cos\theta) + \cos\theta$

$$= D(-\sin\theta) + \cos\theta = -\cos\theta + \cos\theta = 0.$$

Thus if $\alpha \in V$, then α is a linear combination of $\sin\theta$ and $\cos\theta$.

$\therefore \quad (D^2 + I)\alpha = O \quad \forall \alpha \in V$.

Here $D^2 + I$ is the zero operator.

Example 18. *Let T be a linear operator on R^2 which is represented in the standard ordered basis by the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Find the characteristic polynomial for T and verify Cayley-Hamilton Theorem.

Sol. The standard ordered basis of R^2 is $B = \{(1, 0), (0, 1)\}$.

First of all, let us obtain Linear Operator T on R^2 whose matrix w.r.t. basis B is A.

$$\left.\begin{aligned} \text{By def., } T(1, 0) &= 0(1, 0) + 2(0, 1) = (0, 1) \\ \text{and} \quad T(0, 1) &= -1(1, 0) + 0(0, 1) = (-1, 0) \end{aligned}\right\}$$

Since $(x, y) = x(1, 0) + y(0, 1)$.

$\therefore \quad T(x, y) = xT(1, 0) + yT(0, 1) = x(0, 1) + y(-1, 0) = (0, x) + (-y, 0) = (-y, x)$.

Thus linear operator T on R^2 is defined as

$T(x, y) = (-y, x) \quad \forall x, y \in R^2$.

$\therefore$ Characteristic polynomial of T is $\det[tI - A]$

$$= \det \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix} = t^2 + 1.$$

Now, we shall show that T satisfies characteristic polynomial *i.e.*,

To prove. $T^2 + I = O$.

Now $\forall (x, y) \in R^2$,

$$\begin{aligned} (T^2 + I)(x, y) &= T^2(x, y) + I(x, y) = T[T(x, y)] + (x, y) = T(-y, x) + (x, y) \\ &= (-x, -y) + (x, y) = (-x + x, -y + y) = (0, 0) \end{aligned}$$

$\Rightarrow \quad (T^2 + I)(x, y) = (0, 0) \quad \forall x, y \in R^2$

Hence $\quad T^2 + I = O$ is the zero operator.

Example 19. *Prove that a matrix and its transpose A^t have the same characteristic polynomial.* **(P.U. 1998)**

Sol. We know that the matrix and its transpose have same determinant.

$$\begin{aligned} \therefore \quad \det[tI - A] &= \det[(tI - A)^t] = \det[(tI)^t - A^t] \\ &= \det[tI - A^t] \qquad [\because (tI)^t = t(I)^t = tI] \end{aligned}$$

Hence A and A^t have same characteristic polynomial.

Example 20. *Prove that the minimal polynomial of a matrix exists uniquely.*

Sol. Let A be the given matrix.

A is a zero of some non-zero polynomial. [*By Cayley-Hamilton Th.*]

Let n be the lowest degree for which $f(t)$ exists such that $f(A) = O$.

Now dividing $f(t)$ by its leading term, we get a monic polynomial $m(t)$ having degree n which has A as a zero.

$\Rightarrow \quad m(t)$ is minimal polynomial of A.

Uniqueness. Let $m'(t)$ be another monic polynomial of degree n for which $m'(A) = O$

$\Rightarrow$ $m(t) - m'(t)$ is a non-zero polynomial of degree less than n which has A as a zero.

This leads to contradiction.

Hence the minimal polynomial $m(t)$ is unique.

Example 21. *Let T, a linear operator on a vector space V of dimension n, be invertible, then prove that T^{-1} is the polynomial in T of degree not more than n.*

Sol. Let $m(t)$ be the minimal polynomial T.

Then $m(t) = a_0 + a_1 t + \ldots\ldots + a_{r-1} t^{r-1} + t^r$, where $r \le n$.

Since t is invertible, $a_0 \ne 0$,

$\therefore$ $m(t) = a_0 I + a_1 T + \ldots\ldots + a_{r-1} T^{r-1} + T^r = O$

$$\Rightarrow \quad I = -\frac{1}{a_0}(a_1 I + \ldots\ldots + a_{r-1} T^{r-2} + T^{r-1})\, T.$$

Multiplying by T^{-1}, $T^{-1} = -\dfrac{1}{a_0}(a_1 I + \ldots\ldots + a_{r-1} T^{r-2} + T^{r-1})$ [*Using* $TT^{-1} = I$]

Hence T^{-1} is a polynomial in T of degree $r - 1$, which is

$\le n - 1 < n$. [$\because r \le n$]

Example 22. *Let A be a 3 × 3 matrix over T. Show that A cannot be a zero of the polynomial $f(t) = t^2 + 1$.*

Sol. By Cayley-Hamilton Theorem, A is a zero of its characteristic polynomial $\Delta(t)$.

Since $\Delta(t)$ is a polynomial of degree 3,

$\therefore$ it must have at least one real zero.

Let us assume that A is zero of $f(t)$.

Since $f(t)$ is irreducible over R,

$\therefore$ $f(t)$ must be the minimal polynomial of A. [*Def.*]

But $f(t)$ has no real zero.

This leads to contradict the fact that the characteristic and minimal polynomials have same roots.

Hence A is not a zero of $f(t)$.

Example 23. *Let T be a linear operator on a vector space V of finite dimension. Show that T is invertible iff the constant term of the minimal (characteristic) polynomial of T is not zero.*

Sol. Let the minimal (characteristic) polynomial of T be

$$f(t) = a_0 + a_1 t + \ldots\ldots + a_{r-1} t^{r-1} + a_r t^r.$$

Now T is invertible

iff T is non-singular

iff zero is not an eigen value of T

iff zero is not a root of minimal (characteristic) polynomial of T

iff the constant term a_0 is not zero.

Hence the result.

5

SPECIAL TYPES OF MATRICES

1. Symmetric and Skew-Symmetric Matrices

(*i*) **Symmetric Matrix.** *A square matrix* $A = [a_{ij}]$ *is said to be symmetric if* $a_{ij} = a_{ji}$ *for all values of i and j* (*i.e.* (*i, j*)*th element equals* (*j, i*)*th element*).

Thus matrix A is symmetric if $A' = A$.

For examples :

$$\begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

are all symmetric matrices.

Theorem. *The necessary and sufficient condition for the matrix A to be symmetric is that* $A' = A$.

The condition is necessary.

Let $A = [a_{ij}]$ be *n*-rowed symmetric matrix

$\therefore$ by def., $a_{ij} = a_{ji}$ ...(1)

Also A' will be *n*-rowed square matrix. ...(2)

Now (*i, j*)th element of A' = (*j, i*)th element of $A = a_{ji} = a_{ij}$ [*Using* (*1*)]

= (*i, j*)th element of A

(2) and (3) $\Rightarrow$ $A' = A$.

The condition is sufficient.

Here $A' = A$ [*Given*]

$\Rightarrow$ A is a square matrix.

Also (*i, j*)th element of A = (*i, j*)th element of A' [$\because A = A'$]

= (*j, i*)th element of A.

Hence A is a symmetric matrix.

(*ii*) **Skew-Symmetric Matrix.** *A square matrix* $A = [a_{ij}]$ *is said to be skew-symmetric if* $a_{ij} = -a_{ji}$ *for all values of i and j* (*i.e.* (*i, j*)*th element if – ve of* (*j, i*)*th element*).

Since the diagonal elements are of the type $a_{11}, a_{22}, \ldots\ldots, a_{ii}$ and by the given condition $a_{ii} = -a_{ii}$ for all *i*

$\Rightarrow$ $2a_{ii} = 0$ $\Rightarrow$ $a_{ii} = 0$.

Hence the diagonal elements of a skew-symmetric matrix are zero.

For example :

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$ is a skew-symmetric matrix.